PROBLEMS GIVEN AT THE ST. ANDREWS COLLOQUIUM, 1996

First Day: Number Theory

1. Somebody incorrectly remembered Fermat's little theorem as saying that the congruence $a^{n+1} \equiv a \pmod{n}$ holds for all a if n is prime. Describe the set of integers n for which this property is in fact true.

2. Prove or disprove: for every odd number k there is a prime of the form $2^n k+1$.

3. Show that if $(a^2+b^2)/(ab+1)$ $(a,\, b \in \mathbb{N})$ is integral, then it is a perfect square.

Second Day: Algebra

1. Show that there is no ring (=commutative, with 1) with exactly five units.

2. Show that if a group has only finitely many elements of finite order, then these form a subgroup.

3. Let P and S denote the direct product and direct sum of countably many copies of $\mathbb{Z}$, respectively, i.e. P is the set of all sequences of integers $(a_1, a_2, a_3, \dots)$, with the abelian group structure given by componentwise addition, and S is the subgroup consisting of all sequences with $a_i = 0$ for $i \gg 0$. Show that $\mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z}) \approx S$.

Third Day: Polynomials

1. Associate to a prime the polynomial whose coefficients are the decimal digits of the prime (e.g. $9x^3 + 4x^2 + 3$ for the prime 9403). Show that this polynomial is always irreducible.

2. Suppose that $\sum_{n\geq 0} b_n x^n \in \mathbb{R}[x]$ has only real roots. Show that the polynomial $\sum_{n\geq 0} b_n x^n/n!$ has the same property.

3. Let $p = 2m-1$ be an odd prime. Show that the polynomial $(1-x)^m + 1 + x^m$ is twice a square in $\mathbb{F}_p[x]$.

Fourth Day: Geometry

1. Let A and B be two *adjacent* vertices of an equilateral polygon. If the angles at all other vertices are known to be rational (when measured in degrees), show that the angles at A and B are also rational. Give a counterexample to this statement when A and B are not assumed to be adjacent.

2. A rectangle R is the union of finitely many smaller rectangles (non-overlapping except on their boundaries), each one of which has at least one rational side. Show that R has the same property.

3. Mark an angle α ($0 < \alpha < 2\pi$) on a pie-plate, and pick another angle β ($0 < \beta < 2\pi$). Define an operation on pie as follows: cut out the slice of pie over the marked angle, lift it up, turn it over, replace it, and rotate the whole pie on the plate by the angle β. Show that, whatever the values of α and β, this operation has finite order (i.e., after a finite number of iterations every piece of the pie is in its original position).

Fifth Day: Sequences and recursions

1. Define a sequence $\{t_n\}_{n\geq 1}$ by the recursion $t_{n+5} = \big(t_{n+4}t_{n+1} + t_{n+3}t_{n+2}\big)/t_n$, with initial values $(1, 1, 1, 1, 1)$. Prove the statement that all of the t_n are integral.

2. Define a sequence $\{u_n\}_{n\geq 1}$ by the formula $u_n = \sum\limits_{k=1}^{n} \dfrac{(2n+k)!}{(2n-2k)!\,(3k+1)!}$. Show that the statement "none of the u_n are integral" is false, but that the first counterexample is approximately $10^{2019025}$.

3. Define a sequence $\{v_n\}_{n\geq 1}$ by the recursion $v_n = (2 + v_1^2 + \cdots + v_{n-1}^2)/n$. Show that the statement "all of the v_n are integral" is false, but that the first counterexample is approximately $10^{178485291567}$.