Engel Elements in Groups

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Groups St Andrews - 13th August 2009

Left Normed Engel Commutators

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$$[x,_n y] = \big[[x,_{n-1} y], y\big] \quad \text{for all} \quad n > 0.$$

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A “good” fact

$$[_n y, x] = [x,_n y^{-1}]^{y^n}.$$

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If $g^{-1} \in R(G)$, then $\left(g^{-1}\right)^x \in R(G)$ for any $x \in G$ (all Engel subsets are invariants under automorphisms) and so $\left[\left(g^{-1}\right)^x,_n g\right] = 1$ for some $n \geq 1$ which implies that $[x,_{n+1} g] = 1$.

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and so $\big[\big(g^{-1}\big)^x,_n g\big] = 1$ for some $n \geq 1$ which implies that
$[x,_{n+1} g] = 1$. Hence $R(G)^{-1} \subseteq L(G)$ or equivalently
$R(G) \subseteq L(G)^{-1}$.

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Question D. *For which integer* $n \geq 1$, $R_n(G) \subseteq \overline{L}(G)$ *for any group* G*?*

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Theorem (Macdonald 1970) *For any prime number p and each multiple $n > 2$ of p, there is a finite metabelian p-group G containing an element $a \in R_n(G)$ such that $a \notin L_n(G)$ and $a^{-1} \notin L_n(G)$*

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$R_n \nsubseteq L_n \cup L_n^{-1}$–continued

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Theorem (Newman & Nickel 1994) *For every $n \geq 5$ there is a group which*
– has a right n-Engel element a and an element b such that the commutator $[b,_n a]$ has infinite order and in particular, no non-trivial power of a is a left n-Engel element,
– is nilpotent of class $n+2$,
– is abelian-by-(nilpotent of class 3)
– is (nilpotent of class 2)-by-cyclic.

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• (Bludov 2005) There exists a group containing two left Engel elements whose product is not a left Engel element.

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Problem. Prove that the free 2-generated Burnside group of exponent 2^{48} is not a bounded Engel group.

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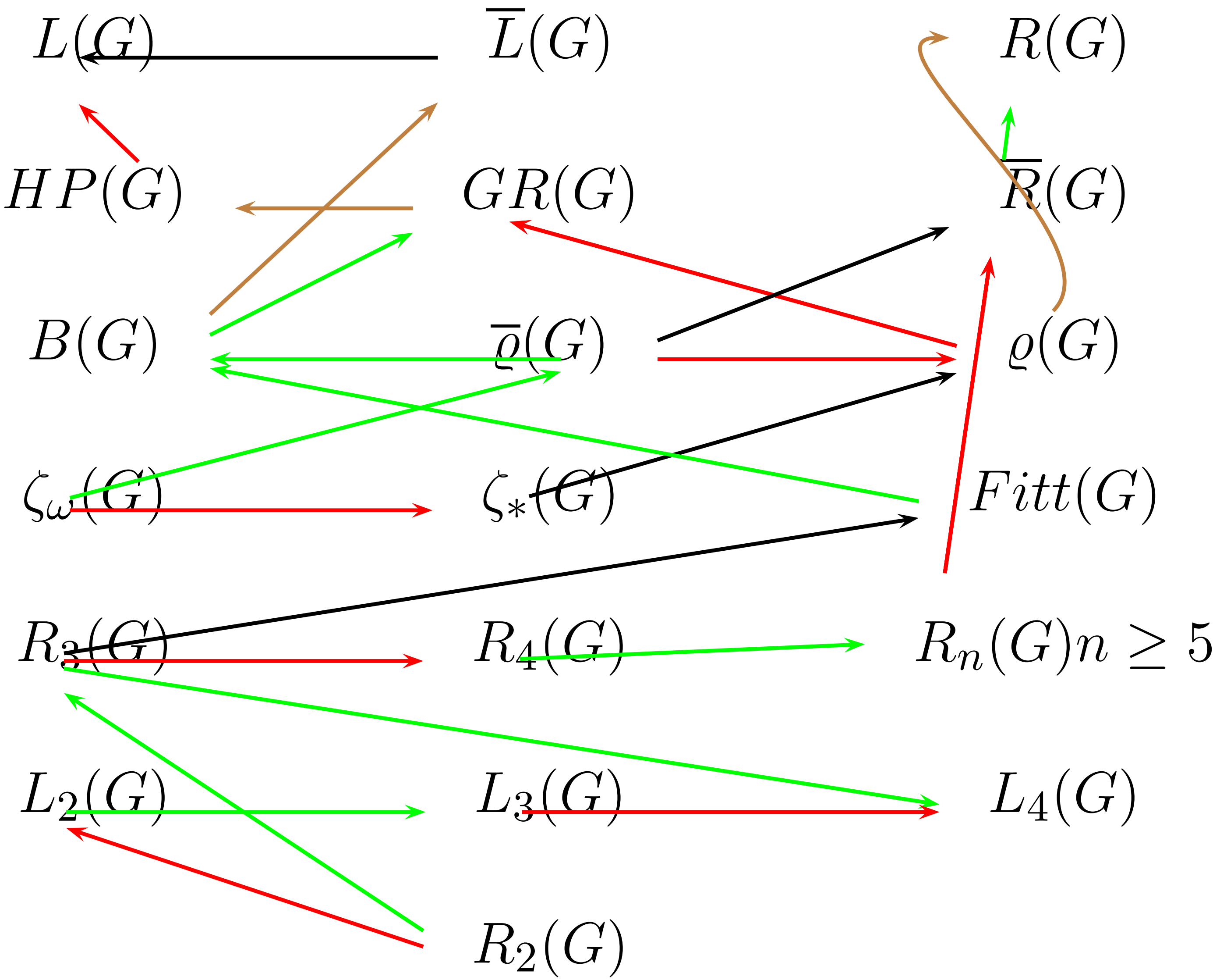
Subgroups and Engel sets



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• **(Gruenberg)** If G is a soluble group, then $L(G) = HP(G)$, $\overline{L}(G) = B(G)$, $R(G) = \varrho(G)$ and $\overline{R}(G) = \overline{\varrho}(G)$.

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• (A & Khosravi) Let G be a group such that $\gamma_5(G)$ has no element of order 2. Then $R_3(G)$ is a subgroup of G. In particular, the set of right 3-Engel elements forms a subgroup in any group without elements of order 2.

Right 4-Engel Elements-I

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- (A & Khosravi) Let G be any group. If $a \in G$ and both $b, b^{-1} \in R_4(G)$, then $\langle a, a^b \rangle$ is nilpotent of class at most 4.

Right 4-Engel Elements-II

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- (A & Khosravi) Let G be a $\{2,3,5\}$-free group such that $\langle a,b,x\rangle$ is nilpotent for all $a,b \in R_4(G)$ and for any $x \in G$. Then $R_4(G)$ is a nilpotent group of class at most 7. In particular, the normal closure of every right 4-Engel element of G is nilpotent of class at most 7.

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- **Question.** What is the least positive integer n such that $L_n(G) \not\subseteq HP(G)$ for some group G? $3 \leq n \leq 13$

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- (A) For an arbitrary group G, $L_3(G)$ consists of elements $x \in G$ such that $\langle x, x^y \rangle$ is nilpotent of class at most 2 for all $y \in G$. In particular, every power of a left 3-Engel element is also a left 3-Engel element and $(G)L_3 = L_3(G)$.

Let 3-Engel Elements-II

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- (A) Let p be any prime number and G be a group. If $x \in L_3(G)$ and $x^{p^n} = 1$ for some integer $n > 1$, then $\langle x^p \rangle^G$ is soluble of derived length at most $n-1$ and $x^p \in B(G)$. In particular, $\langle x^p \rangle^G$ is locally nilpotent.

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• (A) Let G be a group. If both a and a^{-1} belong to $L_4(G)$ and a is a p-element for some prime p, then (1) If $p = 2$ then $a^4 \in B(G)$.
(2) If p is an odd prime, then $a^p \in B(G)$.

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• **Question.** Are there positive integers m, n such that $a^n \in \zeta_m(G)$ for any $a \in R_3(G)$ and for any group G?

An Application of Zel′manov's Theorems in Engel theory

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• **Question.** Are there functions $c, e : \mathbb{N} \to \mathbb{N}$ such that for any nilpotent group G and any normal subgroup H of G with $H \subseteq R_n(G)$, we have $H^{e(n)} \subseteq \zeta_{c(n)}(G)$?