Positive laws on large sets of generators

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Positive words and laws

- Let X be an alphabet of symbols $\{x_1, x_2, \ldots\}$; a word on X is an element of the free group on X. If the word uses n different symbols, we write $w = w(x_1, \ldots, x_n)$.
- The word w is positive if it does not involve any inverses of the symbols x_i.

Let T be a subset of a group G. If α, β are two different positive words on $\{x_1, x_2, \ldots, x_n\}$, T satisfies the positive law

$$\alpha(x_1, \ldots, x_n) = \beta(x_1, \ldots, x_n)$$

if every substitution $x_i \mapsto t_i$ with $t_i \in T$ gives the same value for α and β. The degree of the law is $\max\{|\alpha|, |\beta|\}$. If α and β have the same length the law is homogeneous.

Examples

- A group of finite exponent e satisfies the non-homogeneous exponent law $x^e = 1$.
- An abelian group satisfies the non-positive law $x^{-1}y^{-1}xy = 1$, but also the positive law $xy = yx$.

Malcev-Thue-Morse laws

Define recursively two series of positive words on $\{x, y\}$:

$$\begin{cases} \alpha_0 = x, & \beta_0 = y, \\ \alpha_c = \alpha_{c-1}\beta_{c-1}, & \beta_c = \beta_{c-1}\alpha_{c-1}, \quad (c \geq 1). \end{cases}$$

$M_c(x, y) : \alpha_c(x, y) = \beta_c(x, y)$ is a homogeneous positive law of degree 2^c. For example, $M_1(x, y)$ is $xy = yx$ and $M_2(x, y)$ is $xyyx = yxxy$.

Properties of $M_c(x,y)$

- If $G/Z(G)$ satisfies the positive law $\alpha = \beta$ then G satisfies the positive law $\alpha\beta = \beta\alpha$.
- If N satisfies a positive law and G/N satisfies an exponent law then also G satisfies a positive law.
- If G/N satisfies the exponent law $x^2 = 1$ and N satisfies $M_c(x,y)$ then G satisfies $M_{c+2}(x,y)$.
  (Note $M_{c+1}(x,y) = M_c(xy, yx)$).

- All nilpotent groups of class c satisfy the same positive law $M_c(x,y)$ in two variables.
- Nilpotent-by-(finite exponent) groups satisfy the positive law $M_c(x^e, y^e)$ for some c, e.
- An extension of a nilpotent group by a 2-group satisfies $M_c(x,y)$ for some c.

Properties of $M_c(x,y)$ II

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An extension of a nilpotent group by a 2-group satisfies $M_c(x,y)$ for some c.

Does the converse of the last result hold?

Theorem (Shirshov (1963) - Boffa, Point (1991))

If G is a finite group satisfying $M_c(x,y)$ then G is an extension of a nilpotent group by a 2-group of exponent $\leq 2^{c-2}$.

Does a positive law imply nilpotent-by-(finite exponent)?

Recall: A nilpotent-by-(finite exponent) group satisfies a Malcev law.

(1953) Conjecture:

Every group G, satisfying a positive law, is nilpotent-by-(finite exponent).

Negative general answer

(1996) Olshanskii, Storozhev : 2-generated counterexample.

But positive answer for many classes of groups!

- (2003)Burns, Medvedev and (2004)Bajorska, Macedońska: For the large class of all locally graded groups (i.e. groups in which every non-trivial finitely generated subgroup has a proper subgroup of finite index),the group is nilpotent-by-(locally finite of finite exponent), with n-bounded class and exponent, if n is the degree of the law.

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General problem

Question

Let T be a set of generators of G. If T satisfies a positive law does this imply that the whole group G satisfies a (maybe different) positive law?

- General negative answer.

Example

Let $P = \langle R \rangle$ and $Q = \langle S \rangle$ be two finite p-groups. The free product $G = P * Q$ is a residually-p group generated by the set $U = R \cup S$. U satisfies an exponent law but G is not nilpotent-by-finite (unless P and Q have order 2).

- The size of T plays a fundamental role in the problem. Maybe we have positive answer if T is sufficiently large?
- How "large" should the set of generators be?

Positive laws on commutators

Question

If the set of all commutators in G satisfies a positive law, does it follow that the derived subgroup G' also satisfies a positive law?

Theorem (Fernández-Alcober, Shumyatsky (2007))

Let G be a finitely generated residually-p group. If all simple commutators of length m in G satisfy a positive law of degree n and $p \notin P(n)$, then $\gamma_m(G)$ satisfies a positive law.

Note:

- $P(n)$ is a finite set of "bad primes" which depends on n.
- The set of commutators is a "large" set of generators of G'; it is
  - a normal subset,
  - a commutator-closed subset (closed under taking commutators of its elements).

A generalization: residually-p case

Question

If G is a f.g. residually-p group and T is a commutator-closed normal set of generators of G satisfying a positive law, does also G satisfy a positive law?

Why residually-p?

The residually-p case allows a twofold analysis:

- Via finite p-groups, by approximating the group with its finite quotients.
- Via pro-p groups, by embedding the group in its pro-p completion. This makes it possible to use very powerful tools.

A generalization: positive law on the whole group G

Theorem (Fernández-Alcober, Shumyatsky (2007))

Let G be a f.g. residually-p group and let T be a commutator-closed normal set of generators of G satisfying a positive law of degree n. If G also satisfies a certain law $v = 1$ then:

- If $p \notin P(n)$ then G is nilpotent of bounded class.

Thus the whole group G satisfies a positive law whose length is bounded.

Remark The set $P(n)$ is a real obstruction. For every prime p there exists a 2-generated metabelian residually-p group which is not nilpotent-by-finite but can be generated by a commutator-closed normal subset satisfying a positive law. Note that the law depends on the prime p.

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Fixing the law

In the previous example, for every prime p, we have:

- T is normal and commutator closed.
- T satisfies a positive law depending on p.
- G is a metabelian finitely (2-)generated residually-p group.
- G does not satisfy a positive law.

Now suppose that we:

- Fix the law $M_c(x,y)$.
- Take T normal, commutator-closed and satisfying $M_c(x,y)$.
- Eliminate the finitely generated condition for G.

Question

What will happen? Can we construct an example where G does not satisfy a positive law?

The result

Theorem (Fernández-Alcober, Gavioli, A.)

For each $c \geq 3$ there exists an (infinitely generated) metabelian group G such that:

- *G is residually-p for all primes p.*
- *G can be generated by a commutator-closed normal subset T satisfying the positive law $M_c(x, y)$.*
- *G does not satisfy a positive law.*

How can we construct the group G?

Fix $c \geq 3$.

We want to construct an infinite direct product

$$G = G_c \times G_{c+1} \times \cdots \times G_n \times \cdots$$

where every group G_n has these properties:

- G_n is a nilpotent residually-p group.
- G_n can be generated by a normal commutator-closed subset T_n which satisfies the law $M_c(x,y)$.
- G_n does not satisfy $M_n(x,y)$.

Remark: Note that $n \geq c$ and the "distance" between $M_c(x,y)$ and $M_n(x,y)$ increases with n. From this it follows that G cannot satisfy any law $M_n(x,y)$.

Idea of the proof

How can we construct every group G_n?

We take $G_n = B_n \ltimes A_n$ where:

- $B_n = \langle t_1, \dots, t_n \rangle$ is a group of matrices of size $d = d(n)$ and $t_1, \dots, t_n$ commute.
- $A_n = \mathbb{Z} \times \overset{d(n)}{\cdots} \times \mathbb{Z}$.

Which is the subset T_n?

We put $T_n = t_1 A_n \cup \cdots \cup t_n A_n \cup A_n$. Then

- T_n is commutator-closed and normal;
- T_n generates all of G_n.

We are going to choose the matrices $t_1, \dots, t_n$ such that:

- T_n satisfies the law $M_c(x,y)$.
- G_n does not satisfy $M_n(x,y)$.

T_n satisfies $M_c(x, y)$ and G_n does not satisfy $M_n(x, y)$

Remark Since $M_c(x, y)$ depends only on two variables, it suffices to require that $tA_n \cup uA_n \cup A_n$ satisfies the law, where $t, u \in \{t_1, \ldots, t_n\}$.

Consequences of a technical lemma

In the group G_n that we want to construct, T_n satisfies $M_c(x, y)$ if and only if

- $(t_1 - 1)^c = \cdots = (t_n - 1)^c = 0$;
- $(t_i - 1)(t_i t_j - 1)^{c-1} = 0$ for each i, j.

Furthermore G_n does not satisfy $M_n(x, y)$ provided that

- $(t_1 \ldots t_n - 1)^n \neq 0$.

In order to guarantee the three items above

- We work inside $A = \mathbb{Q}[X_1, \ldots, X_n]/\mathfrak{a}$, with $\dim(A) = d$, where $\mathfrak{a}$ is an appropriate monomial ideal.
- Then we consider the regular representation of A : $A \xrightarrow{\varphi} \mathcal{M}_d(\mathbb{Q})$ and we define $t_i = \varphi(\overline{X_i + 1}),\ i \leq n$.

G does not satisfy a positive law

Finally we consider

$$G_n = \langle t_1, \dots, t_n \rangle \ltimes (\mathbb{Z} \times \overset{d}{\cdots} \times \mathbb{Z}).$$

- G_n is residually-p for every prime p because it is a finitely generated torsion-free nilpotent group.

Recall that $G = G_c \times G_{c+1} \times \cdots \times G_n \times \cdots$, so

- G is metabelian and residually-p for every prime p.
- G can be generated by $T = \bigcup_{n \geq c} T_n$.
- Note that T is normal commutator-closed and it satisfies $M_c(x,y)$.
- By way of contradiction we conclude that the whole group G does not satisfy a positive law.

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A question about positive laws on words and verbal subgroups

Question

If w is a word, write G_w for the set of all values of w in a group G and their inverses, and $w(G) = \langle G_w \rangle$ for the verbal subgroup. Does a positive law on G_w imply a positive law on the whole of $w(G)$?

Recall that: A word w has finite width in G if every element of $w(G)$ can be written as a product of at most k elements of G_w for some fixed k.

Theorem (Jaikin-Zapirain (2008))

Let G be a p-adic analytic pro-p group. Then all words have finite width in G.

The case of p-adic analytic groups

Theorem (Fernández-Alcober, Shumyatsky (2007))

Let G be a p-adic analytic pro-p group and let w be any commutator-closed word. If G_w satisfies a positive law of degree n and $p \notin P(n)$, where $P(n)$ is a finite set of "bad primes", then the verbal subgroup $w(G)$ satisfies a positive law.

Open problems

- Note that not all words are commutator-closed. Is it possible to generalize this result to all words w?
- This could be done if we obtain analogous results to those of Fernández-Alcober and Shumyatsky for a normal commutator-closed generating set T but only with the assumption that T is normal.

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Thank you !