A survey of recent progress on nonabelian tensor squares of groups

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L.-C. Kappe: "Nonabelian tensor products of groups: the commutator connection"

- report on developments since the 1987 paper "Some computations of non-abelian tensor products of groups" by Brown, Johnson and Robertson
- conceptual connection between commutators and non-abelian tensors
- status of questions posed in Brown, Johnson and Robertson
- an explicit invitation to join investigations of non-abelian tensor products
- notation: "driving on the left"

  conjugation on the left: ${}^{h}g = hgh^{-1}$
  commutator: $[h, g] = hgh^{-1}g^{-1}$

Definition of a nonabelian tensor product

Definition

Two groups G and H *act compatibly* on each other if

$$^{(^g h)}g' = {}^g({}^h({}^{g^{-1}}g')) \quad \text{and} \quad {}^{(^h g)}h' = {}^h({}^g({}^{h^{-1}}h'))$$

for all $g, g' \in G, h, h' \in H$.

Definition

Let G and H be two groups that act compatibly on each other and on themselves by conjugation. The *nonabelian tensor product* $G \otimes H$ is the group generated by the symbols $g \otimes h$, where $g \in G, h \in H$, subject to the relations

$$gg' \otimes h = ({}^g g' \otimes {}^g h)(g \otimes h) \quad \text{and} \quad g \otimes hh' = (g \otimes h)({}^h g \otimes {}^h h')$$

for all $g, g' \in G$ and all $h, h' \in H$.
If $G = H$ and all actions are conjugation, then $G \otimes G$ is called the *non-abelian tensor square of* G.

Origins of the non-abelian tensor product

Foundations:

J.H.C. Whitehead. A certain exact sequence, *Ann. of Math. (2)* **52** (1950), 51–110.

C. Miller. The second homology group of a group; relations among commutators, *Proc. Amer. Math. Soc.* **3** (1952), 588–595.

R.K. Dennis. In search of "new" homology functors having a close relationship to $K-$theory, Preprint, Cornell University, Ithaca, NY 1976.

First definitions:

R. Brown, J.L. Loday. Excision homotopique en basse dimension., *C.R. Acad. Sci. Pars Sér. I Math* **298** (1984), no. 15, 353–356.

R. Brown, J.L. Loday. Van Kampen theorems for diagram of spaces., *Topology* **26** (1987), no. 3, 311–335. With an appendix by M. Zisman.

As group theoretic objects:

R. Brown, D.L. Johnson, E.F. Robertson. Some computations of nonabelian tensor products of groups, *J. Algebra* **111** (1987), no. 1, 177–202.

R. Aboughazi. Produit tensoriel du group d'Heisenberg, *Bull. Soc. Math. France* **115** (1987), 95–106.

G.J. Ellis. The nonabelian tensor product of finite groups is finite, *J. Algebra* **111** (1987), 203–205.

D.L. Johnson. The nonabelian tensor square of a finite split metacyclic group, *Proc. Edinburgh Math. Soc.* **30** (1987), 91–96.

Definitions of the nonabelian exterior square, κ and κ'

Let $\nabla(G) = \langle g \otimes g \mid g \in G \rangle \leq G \otimes G$.

$\nabla(G) \leq Z(G \otimes G)$.

Definition

The *nonabelian exterior square* $G \wedge G$ of the group G is the factor group $G \otimes G / \nabla(G)$

Epimorphisms:

Let $\kappa : G \otimes G \longrightarrow G'$ be defined by $\kappa(g \otimes h) = [g, h]$ for all $g, h \in G$.

Let $\kappa' : G \wedge G \longrightarrow G'$ be the induced map.

Commutative diagram from Brown and Loday (1987)

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
H_3(G) & \longrightarrow & \Gamma(G^{\mathrm{ab}}) & \xrightarrow{\psi} & J_2(G) & \longrightarrow & H_2(G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \nabla(G) & \longrightarrow & G \otimes G & \longrightarrow & G \wedge G & \longrightarrow & 1 \\
 & & & & {\scriptstyle\kappa}\downarrow & & {\scriptstyle\kappa'}\downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

where all the sequences are exact and the short exact sequences are central. $\Gamma(G^{\mathrm{ab}})$ is the Whitehead quadratic functor.

Results of Brown, Johnson and Robertson (1987)

Proposition (also Ellis (1987))

If G is a finite group, then so is $G \otimes G$.
If G is a finite $p-$group, then so is $G \otimes G$.

Proposition

If G is a free group, then $G \otimes G \cong G' \times \Gamma(G^{\mathrm{ab}})$.

If G is free of finite rank $n \geq 2$, then G' is free of countably infinite rank and $\Gamma(G^{\mathrm{ab}})$ is free abelian of rank $\frac{n(n+1)}{2}$.

Computing non-abelian tensor squares

Using the definition

By "computing" a non-abelian tensor square we mean that the non-abelian tensor square is identified in a generally recognized structural form or presentation.

Brown, Johnson and Robertson (1987) computed the non-abelian tensor squares of all groups of order up to 30 using the defining generators and relations given by the definition.

This method quickly becomes untenable for larger finite groups, since the definition provides $|G|^2$ generators and $2|G|^3$ relators.

Computing non-abelian tensor squares

Crossed pairings: definition and method

Definition

Let G, H and L be groups. A function $\phi : G \times H \to L$ is called a *crossed pairing* if for all $g, g' \in G$ and all $h, h' \in H$,

$$\phi(gg', h) = \phi({}^g g', {}^g h)\phi(g, h),$$

$$\phi(g, hh') = \phi(g, h)\phi({}^h g, {}^h h').$$

A crossed pairing determines a unique homomorphism of groups $\phi^* : G \otimes H \to L$ so that $\phi^*(g \otimes h) = \phi(g, h)$ for all $g \in G$ and $h \in H$.

Method: Conjecture a group L for $G \otimes G$, as well as a map $\phi : G \times G \to L$. Show that ϕ is a crossed pairing and that the induced map ϕ^* is an isomorphism.

Computing non-abelian tensor squares

Crossed pairings: sample result

Theorem (Bacon, Kappe and Morse (1997); Blyth, Morse and Redden (2004))

(i) *The non-abelian tensor square of the free* $2-$*Engel group of rank* 2 *is free abelian of rank* 6*.*

(ii) *The non-abelian tensor square of the free* $2-$*Engel group of rank* $n > 2$ *is a direct product of a free abelian group of rank* $\frac{1}{3}n(n^2+2)$ *and an* $n(n-1)-$*generated nilpotent group of class* 2 *whose derived subgroup has exponent* 3*.*

Very detailed computer-assisted calculations, sufficient to dissuade attempting to use crossed pairings to investigate non-abelian tensor squares of (e.g.) finite rank free nilpotent groups of class 3.

Computing non-abelian tensor squares

The group $\nu(G)$

Definition (Ellis and Leonard (1995), Rocco (1991))

Let G be a group with presentation $\langle \mathcal{G} | \mathcal{R} \rangle$ and let G^{φ} be an isomorphic copy of G via the mapping $\varphi : g \to g^{\varphi}$ for all $g \in G$. Define the group $\nu(G)$ to be

$$\nu(G) = \langle \mathcal{G}, \mathcal{G}^{\varphi} | \mathcal{R}, \mathcal{R}^{\varphi}, {}^{x}[g, h^{\varphi}] = [{}^{x}g, ({}^{x}h)^{\varphi}] = {}^{x^{\varphi}}[g, h^{\varphi}], \forall x, g, h \in G \rangle.$$

The groups G and G^{φ} embed isomorphically into $\nu(G)$. By convention the labels G and G^{φ} also denote their natural isomorphic copies in $\nu(G)$.

Theorem (Ellis and Leonard (1995), Rocco (1991))

Let G be a group. The map

$$\phi : G \otimes G \to [G, G^{\varphi}] \lhd \nu(G)$$

defined by $\phi(g \otimes h) = [g, h^{\varphi}]$ for all g and h in G is an isomorphism.

McDermott (1998) extends results of Ellis and Leonard (1995) to obtain for a finite group G a simplified presentation for $\nu(G)$ in terms of a generating sequence $\mathcal{L}_G$ relative to a subnormal series for G. From such a presentation one can compute a concrete representation of $\nu(G)$ and thence $[G, G^{\varphi}]$.

Computing non-abelian tensor squares

Properties of $\nu(G)$

Theorem (Rocco (1991))

Let G be a group.

(i) *If G is finite then $\nu(G)$ is finite.*

(ii) *If G is a finite p-group then $\nu(G)$ is a finite p-group.*

(iii) *If G is nilpotent of class c then $\nu(G)$ is nilpotent of class at most $c+1$.*

(iv) *If G is solvable of derived length d then $\nu(G)$ is solvable of derived length at most $d+1$.*

For G nilpotent or soluble, use nilpotent or soluble quotient algorithm to find a concrete presentation of $\nu(G)$.
Implemented by Ellis and Leonard (1995) using CAYLEY, by Ellis (1998) using Magma, and by McDermott (1998) and Rocco (1994) using GAP.

Computing non-abelian tensor squares

Polycyclic groups

Theorem (Blyth and Morse (2009))

Let G be a polycyclic group with a finite presentation $\langle \mathcal{G} \mid \mathcal{R} \rangle$ and polycyclic generating set $\mathfrak{G}$. Then

(i) *The groups $G \otimes G$ and $\nu(G)$ are polycyclic.*

(ii) *The group $\nu(G)$ has a finite presentation that depends only on $\mathcal{G}$, $\mathcal{R}$ and $\mathfrak{G}$.*

(iii) *The nonabelian tensor square $G \otimes G$ is generated by the set*

$$\{\mathfrak{g}^{\pm 1} \otimes \mathfrak{h}^{\pm 1} \mid \textit{ for all } \mathfrak{g}, \mathfrak{h} \textit{ in } \mathfrak{G}\}.$$

These results support hand and computer calculations, for example, using a polycyclic quotient algorithm.

Computing non-abelian tensor squares

Polycyclic groups

In practice, if G is nilpotent then the nilpotent quotient algorithm `nq` can be used to good effect.

Eick and Nickel (2008) have developed another approach to computing the non-abelian tensor and non-abelian exterior squares that exploits the facts that both non-abelian squares are central extensions. Implemented in the `Polycyclic` package, the method can effectively compute these squares for many infinite polycyclic groups, although for nilpotent groups their method tends to take significantly more time than the approach that uses `nq`.

Computing non-abelian tensor squares

Nonabelian tensor squares of free nilpotent groups

Theorem

Let $\mathcal{N}_{n,c}$ denote the free nilpotent group of class c and rank $n > 1$, and denote the free abelian group of rank n by F_n^{ab}.

(i) *(Bacon (1994)) For $c = 2, \mathcal{N}_{n,2} \otimes \mathcal{N}_{n,2} \cong F_{f(n)}^{\mathrm{ab}}$, where*

$$f(n) = \frac{n(n^2 + 2n - 1)}{3}.$$

(ii) *(Blyth and Morse (2008)) For $c = 3, \mathcal{N}_{n,3} \otimes \mathcal{N}_{n,3}$ is the direct product of W_n and $F_{h(n)}^{\mathrm{ab}}$, where W_n is nilpotent of class 2, minimally generated by $n(n-1)$ elements, and*

$$h(n) = \frac{n(3n^3 + 14n^2 - 3n + 10)}{24}.$$

Structure of non-abelian tensor square

Commutative diagram revisited

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
H_3(G) & \longrightarrow & \Gamma(G^{\mathrm{ab}}) & \xrightarrow{\psi} & J_2(G) & \longrightarrow & H_2(G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \nabla(G) & \longrightarrow & G \otimes G & \longrightarrow & G \wedge G & \longrightarrow & 1 \\
 & & & & \kappa \downarrow & & \kappa' \downarrow & & \\
 & & & & G' & = & G' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

It turns out that the middle row splits under fairly general conditions.

Structure of non-abelian tensor square

Lemma

The following technical lemma, an improvement of Proposition 3.3 of Rocco (1994), shows that the structure of $\nabla(G)$ depends on G^{ab}.

Lemma (BFM)

Let G be a group such that G^{ab} is finitely generated. Assume that G^{ab} is the direct product of the cyclic groups $\langle x_i G' \rangle$, for $i = 1, \ldots, s$ and set $E(G)$ to be $\left\langle [x_i, x_j^{\varphi}] | i < j \right\rangle [G', G^{\varphi}]$. Then the following hold.

(i) *$\nabla(G)$ is generated by the elements of the set $\left\{ [x_i, x_i^{\varphi}], [x_i, x_j^{\varphi}][x_j, x_i^{\varphi}] | 1 \leq i < j \leq s \right\}$.*

(ii) *$[G, G^{\varphi}] = \nabla(G) E(G)$.*

Structure of non-abelian tensor square

Splitting Theorem

Theorem (BFM)

Assume that G^{ab} is finitely generated. Then the following hold.

(i) *The restriction $f|_{\nabla(G)} : \nabla(G) \longrightarrow \nabla(G^{\mathrm{ab}})$ of the projection $f : G \longrightarrow G^{\mathrm{ab}}$ onto G^{ab} has kernel $N = E(G) \cap \nabla(G)$. Moreover, N is a central elementary abelian 2-subgroup of $[G, G^{\varphi}]$ of rank at most the 2-rank of G^{ab}.*

(ii) *$[G, G^{\varphi}]/N \simeq \nabla(G^{\mathrm{ab}}) \times (G \wedge G)$.*

(iii) *Suppose either that G^{ab} has no elements of order two or that G' has a complement in G. Then $\nabla(G) \simeq \nabla(G^{\mathrm{ab}})$ and $G \otimes G \simeq \nabla(G) \times (G \wedge G)$.*

Structure of non-abelian tensor square

Commutator relations

Lemma (Rocco (1991, 1994); Blyth, Moravec and Morse (2008))

Let G be any group. The following relations hold in $\nu(G)$.

(i) ${}^{[g_3, g_4^{\varphi}]}[g_1, g_2^{\varphi}] = {}^{[g_3, g_4]}[g_1, g_2^{\varphi}] = {}^{[g_3^{\varphi}, g_4]}[g_1, g_2^{\varphi}]$, *for all* $g_1, g_2, g_3, g_4 \in G$.

(ii) $[g_1^{\varphi}, g_2, g_3] = [g_1, g_2^{\varphi}, g_3] = [g_1, g_2, g_3^{\varphi}] = [g_1^{\varphi}, g_2^{\varphi}, g_3] = [g_1^{\varphi}, g_2, g_3^{\varphi}] = [g_1, g_2^{\varphi}, g_3^{\varphi}]$, *for all* $g_1, g_2, g_3 \in G$.

(iii) $[g_1, [g_2, g_3]^{\varphi}] = [g_2, g_3, g_1^{\varphi}]^{-1}$, *for all* $g_1, g_2, g_3 \in G$.

(iv) $[g, g^{\varphi}]$ *is central in* $\nu(G)$ *for all* $g \in G$.

(v) $[g, g^{\varphi}] = 1$ *for all* $g \in G'$.

(vi) *If* $g_1 \in G'$ *or* $g_2 \in G'$*, then* $[g_1, g_2^{\varphi}]^{-1} = [g_2, g_1^{\varphi}]$.

Structure of non-abelian tensor square

Corollary

Let G be a group such that G^{ab} is a finitely generated group with no elements of order 2. Then $J(G) \cong \Gamma(G^{\mathrm{ab}}) \times M(G)$.

Corollary (Blyth, Moravec, Morse (2008))

Let $G = \mathcal{N}_{n,c}$ be the free nilpotent group of class c and rank $n > 1$. Then $J(G) \cong \Gamma(G^{\mathrm{ab}}) \times M(G)$ is free abelian of rank $\binom{n+1}{2} + M(n, c+1)$, where $M(n, c)$ denotes the number of basic commutators in n symbols of weight c.

Structure of non-abelian tensor square

Brown, Johnson and Robertson (1987) show that if the Schur multiplicator $M(G)$ of G is finitely generated, then $G \wedge G$ is isomorphic to the derived subgroup of any covering group $\hat{G}$ of G.

Definition

A *covering group* $\hat{G}$ of a group G is a central extension

$$1 \longrightarrow M(G) \xrightarrow{\iota} \hat{G} \longrightarrow G \longrightarrow 1,$$

where the image of ι is a subset of $\hat{G}'$

For example, if G is the free nilpotent group $\mathcal{N}_{n,c}$, then $\hat{G} \cong \mathcal{N}_{n,c+1}$, so that $G \wedge G \cong \mathcal{N}'_{n,c+1}$. We recover the main result of Blyth, Moravec and Morse (2008).

Theorem

Let $G = \mathcal{N}_{n,c}$ *be the free nilpotent group of class* c *and rank* $n > 1$. *Then*

$$G \otimes G \cong \nabla(G) \times (G \wedge G) \cong F^{\mathrm{ab}}_{\binom{n+1}{2}} \times \mathcal{N}'_{n,c+1}.$$

Structure of non-abelian tensor square

Exterior square theorem

The proof of the main result of Miller (1952) can be generalized to show:

Theorem (BFM)

Let G be a group and let F be a free group such that $G \cong F/R$ for some normal subgroup R of F. Then $G \wedge G \cong F'/[F, R]$.

The earlier results of Brown, Johnson and Robertson (1987) on the non-abelian tensor squares of free groups of finite rank and of Blyth, Moravec and Morse (2008) for free nilpotent groups of finite rank follow directly from the splitting and exterior square theorems. We also obtain a result for free soluble groups.

Corollary (BFM)

Let F be the free group of finite rank $n > 1$, let d be a natural number, and let $G = F/F^{(d)}$ be the free solvable group $\mathcal{S}_{n,d}$ of derived length d and rank $n > 1$. Then

$$G \otimes G \simeq \mathbb{Z}^{n(n+1)/2} \times F'/[F, F^{(d)}]$$

is an extension of a nilpotent group of class ≤ 3 by a free solvable group of derived length $d-2$ and infinite rank. In particular, if $d = 2$, then $G \otimes G$ is a nilpotent group.

Connection with capability

Definitions

Definition

A group G is said to be *capable* if it is isomorphic to $H/Z(H)$ for some group H, that is, if G is isomorphic to the group of inner automorphisms of some group H.

Definition

The *epicentre* of a group G is the uniquely determined central subgroup of G that is minimal subject to being the image in G of some central extension of G.

A group G is capable if and only if it has trivial epicentre $Z^*(G)$.

Connection with capability

Definition (Ellis (1995))

Let G be a group.

(i) The *non-abelian tensor centre* $Z^{\otimes}(G)$ of G is

$$Z^{\otimes}(G) = \{g \in G | [g, x^{\varphi}] = 1, \forall x \in G\}.$$

(ii) The *non-abelian exterior square* $Z^{\wedge}(G)$ of G is

$$Z^{\wedge}(G) = \{g \in G | [g, x^{\varphi}] \in \nabla(G), \forall x \in G\}.$$

Connection with capability

Ellis (1995, 1998) showed that $Z^{\otimes}(G)$ is a characteristic central subgroup of G and is the largest normal subgroup L of G such that $G \otimes G \simeq G/L \otimes G/L$. The non-abelian exterior centre $Z^{\wedge}(G)$ is a central subgroup of G and is equal to the epicentre $Z^{*}(G)$ of G. In particular, therefore, a group G is capable if and only if $Z^{\wedge}(G) = 1$.

Corollary (BFM)

Let G be any group such that G^{ab} is finitely generated. With the notation above, if $N = 1$ then $Z^{\otimes}(G) = Z^{\wedge}(G) \cap G'$. In particular, the conclusion holds if G is a finite group of odd order.

Connection with capability

Question

Is it always true that

$$N = [Z^{\wedge}(G) \cap G', G^{\varphi}]?$$

A positive answer to the Question will imply, by Proposition 9 of Brown, Johnson and Robertson, that $G \otimes G/N$ is isomorphic to the tensor square of the group G/H, where H is defined to be $Z^{\wedge}(G) \cap G'$.