Outer commutator words are uniformly concise

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A question of Philip Hall

Hall's question

Let ω be a group word, and let G be a group. If ω takes finitely many values in G, is the verbal subgroup $\omega(G)$ finite?

Recall that $\omega(G)$ is the subgroup generated by all values of ω in G.

Definition

We say that ω is concise if the answer to Hall's question is positive for that word. We similarly speak of words which are concise in a class $\mathcal{C}$ of groups.

So Hall's question amounts to asking: are all words concise?

Some concise words

The following words are concise:

- Words lying outside the commutator subgroup of the free group, i.e. such that the sum of the exponents of the indeterminates is non-zero.
  (P. Hall, 1950's?)
- The lower central words $\gamma_i = [x_1, x_2, \ldots, x_i]$.
  (P. Hall, 1950's?)
- The derived words δ_i, defined recursively by $\delta_0 = x_1$ and

$$\delta_i = [\delta_{i-1}(x_1, \ldots, x_{2^{i-1}}), \delta_{i-1}(x_{2^{i-1}+1}, \ldots, x_{2^i})].$$

  (Turner-Smith, 1964)
  The corresponding verbal subgroups are the derived subgroups $G^{(i)}$.

Outer commutator words

Definition

An outer commutator word is a word which is formed by nesting commutators, but using always different indeterminates.

- The lower central words and the derived words are outer commutator words.
- $[[x_1, x_2], [[x_3, x_4], [x_5, x_6]], x_7]$ is an outer commutator word, but the Engel word $[x, y, y, y]$ is not.

Theorem (Jeremy Wilson, 1974)

All outer commutator words are concise.

Some more results

However, not all words are concise: Ivanov (1989) proved that, if $p > 5000$ is a prime and $n > 10^{10}$ is an odd integer, then the word $[[x^{pn}, y^{pn}]^n, y^{pn}]^n$ is not concise.

Theorem (Merzljakov, 1967)

Every word is concise in the class of linear groups.

Theorem (Turner-Smith, 1964)

Every word is concise in the class formed by the residually finite groups all of whose quotients are again residually finite.

It is still an open question whether all words are concise in the class of residually finite groups.

Quantitative conciseness via ultraproducts

Let ω be a concise word. Is there a function f such that, whenever ω takes m values in a group G, we have $|\omega(G)| \leq f(m)$?

One can see that the answer is positive by way of contradiction: assume there exists a family $\{G_n\}_{n\in\mathbb{N}}$ of groups such that

- ω takes at most m values in every G_n.
- $\lim_{n\to\infty} |\omega(G_n)| = \infty$.

Then the ultraproduct U of these groups with respect to a non-principal ultrafilter has at most m values of ω, but $|\omega(U)| = \infty$.

- The ultraproduct is the quotient of the cartesian product $\prod_{n\in\mathbb{N}} G_n$ by the normal subgroup consisting of all tuples (g_n) such that the set $\{n \in \mathbb{N} \mid g_n = 1\}$ belongs to the ultrafilter.
- The existence of non-principal ultraproducts cannot be proved in ZF set theory. It can be obtained from the Axiom of Choice (via Zorn's Lemma), but it is strictly weaker than AC.

Outer commutator words are uniformly concise

Theorem A (F-A, Morigi)

Let ω be an outer commutator word and let G be a group in which ω takes m different values. Then:

- *If G is soluble, $|\omega(G)| \leq 2^{m-1}$.*
- *If G is not soluble, $|\omega(G)| \leq [(m-1)(m-2)]^{m-1}$.*

- Theorem A was essentially obtained by Brazil, Krasilnikov and Shumyatsky for lower central words and for derived words.
- The most important thing in Theorem A is that the bounds are independent of the outer commutator word. This is why we say that outer commutator words are uniformly concise.
- Also, Theorem A does not depend on ultraproducts.

About the bounds

- The bounds in Theorem A are reasonable, but they can very likely be sharpened. We have not addressed this problem yet.
- In the case of $\omega = [x, y]$, we have the following result.

Theorem

Assume that the set of commutators of the group G is finite, of cardinality m. Then:

- *If G is soluble, then $|G'| \leq m^{\frac{1}{2}(5+\log_2 m)}$ (Peter Neumann, Vaughan-Lee, 1977)*
- *For a general group G, we have $|G'| \leq m^{\frac{1}{2}(13+\log_2 m)}$. (Segal, Shalev, 1999)*

Note that, for example,

$$2^{m-1} = m^{(m-1)/\log_2 m}.$$

About the proof of Theorem A

- Our proof of Theorem A does not depend on Wilson's result about the conciseness of outer commutator words.
- Thus, in particular, we provide an alternative proof to Wilson's result (which, we honestly believe, is easier to follow).
- Our proof relies on the following result, which is interesting by itself and does not require the finiteness of the set of ω-values. It was proved by Brazil, Krasilnikov and Shumyatsky for derived words.

Theorem B (F-A, Morigi)

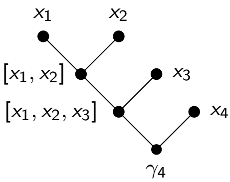
Let ω be an outer commutator word, and let G be a soluble group. Then there exists a series of subgroups from 1 to $\omega(G)$ such that:

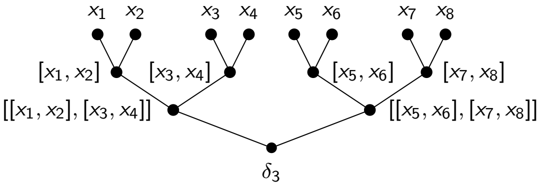
- *All subgroups of the series are normal in G.*
- *Every section of the series is abelian and can be generated by values of ω all of whose powers are again values of ω (in the section).*

We can associate a tree to every outer commutator word ω by recursion:

- If ω is a single indeterminate, then consider an isolated vertex.
- Otherwise, if $\omega = [\alpha, \beta]$, draw the tree of ω by connecting the trees of α and β with a new vertex below them.

The following are the trees of γ_4 and δ_3.





Clearly, we always obtain a dyadic rooted tree.

Height and defect

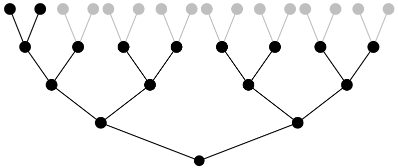
Definition

Let ω be an outer commutator word. Then:

- The height h of ω is the height of its tree.

Observe that the derived word δ_h has height h and its tree is the 'complete' dyadic rooted tree of height h.

- The defect of ω is the number of vertices that we need to add to its tree in order to obtain the tree of δ_h.



$[[\gamma_3, \gamma_2], \delta_2]$

The word $[[\gamma_3, \gamma_2], \delta_2]$ has height 4 and defect 14.

Strategy for the proof of Theorem B

Theorem B

Let ω be an outer commutator word, and G a soluble group. Then there is a series of subgroups from 1 to $\omega(G)$, all normal in G, such that every section of the series is abelian and can be generated by values of ω all of whose powers are values of ω.

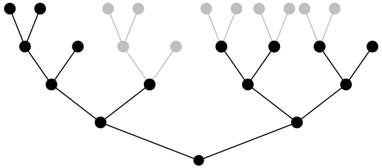
If a series of normal subgroups of G satisfies the last condition, we call it a PCG-series (power-closed generated) w.r.t. ω.
Thus we need to a PCG-series from 1 to $\omega(G)$ w.r.t. ω.

- Argue by induction on the height h and the defect d of ω. If $h = 0$ or $d = 0$ (derived word), the result holds.
- By the induction hypothesis, for every word φ of height h and defect $< d$, there is a PCG-series from 1 to $\varphi(G)$ w.r.t. φ.
- If every value of φ in G is also a value of ω, this is also a PCG-series w.r.t. ω, and we may assume $\varphi(G) = 1$.

Extension of words

Definition

Let φ and ω be two outer commutator words. We say that φ is an extension of ω, if the tree of φ is an upward extension of the tree of ω.



$[\gamma_4, \delta_2]$

An extension of $[\gamma_4, \delta_2]$: $[[\gamma_3, \gamma_3], [\delta_2, \gamma_3]]$.

Making an extension φ of ω corresponds to replacing some indeterminates of ω by other outer commutator words. Hence every value of φ is also a value of ω.

Taking powers out of ω

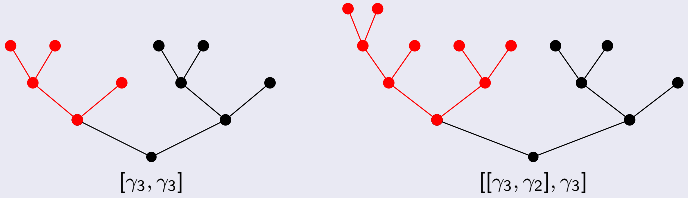
- Let Φ be the (finite) set of all outer commutator words of height h which are a proper extension of ω.
- We may assume that $\varphi(G) = 1$ for every $\varphi \in \Phi$.
- Let now $g \in G$ be a value of ω. If $\omega = [\alpha, \beta]$, we can write $g = [a, b]$, where a is a value of α and b is a value β. Then:
$$g^n \equiv [a^n, b] \pmod{[\omega(G), \alpha(G)]}.$$
- If $[\omega(G), \alpha(G)]$ is contained in $\prod_{\varphi \in \Phi} \varphi(G)$, then $[\omega(G), \alpha(G)] = 1$, and $g^n = [a^n, b]$, and we have taken the power 'inside the first commutator'.
- By applying the induction to α (a word of smaller height), we can take the power 'inside all commutators' and eventually get that g^n is a value of ω. (It is not so simple, but this is the idea.)

The three subgroup lemma

The three subgroup lemma

If $L, M, N \trianglelefteq G$, then $[L, M, N] \leq [M, N, L][N, L, M]$.

- Thus, if $\omega = [\alpha, \beta]$ and φ are outer commutator words,
$$[\omega(G), \varphi(G)] \leq \pi^{(1)}(G)\pi^{(2)}(G),$$
where $\pi^{(1)} = [[\alpha, \varphi], \beta]$ and $\pi^{(2)} = [\alpha, [\beta, \varphi]]$.
- The tree of $\pi^{(1)}$ is very similar to that of ω: replace the tree on top of vertex α with the tree of $[\alpha, \varphi]$. For example:

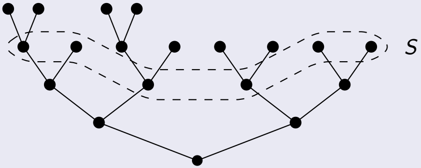


- Thus $\pi^{(1)}$ is obtained from the vertex α and $\pi^{(2)}$ comes from β.

Iterating to sections

Now the idea is to iterate the process of the previous slide in order to reach higher vertices of the tree. Which are the sets of vertices that we get to? In which subgroup will the commutator $[\omega(G), \varphi(G)]$ then be embedded?

By a section we mean a set of vertices which is obtained when we cut the tree from side to side:



A section S of $[[\gamma_3, \gamma_3], \delta_2]$.

The tree subgroup lemma. Conclusion

The tree subgroup lemma

Let ω and φ be two outer commutator words, and let S be a section of the tree T of ω. For every $v \in S$, we define:

- T_v is the subtree of T on top of v, and ω_v is the corresponding word.
- $\pi^{(v)}$ is the word obtained from T by replacing T_v with the tree of $[\omega_v, \varphi]$.

Then $[\omega(G), \varphi(G)] \leq \prod_{v \in S} \pi^{(v)}(G)$.

Now there are a section S of the tree of ω and a word δ_i such that:

- $\alpha(G) \leq \delta_i(G)$, and so $[\omega(G), \alpha(G)] \leq [\omega(G), \delta_i(G)]$.
- If we take $\varphi = \delta_i$ in the Tree Subgroup Lemma, then $\pi^{(v)}(G) \leq \rho^{(v)}(G)$ for some words $\rho^{(v)} \in \Phi$.
- Thus $[\omega(G), \alpha(G)] \leq \prod_{\varphi \in \Phi} \varphi(G)$ and we are done.

The existence of S and δ_i is obtained again from the tree of ω.