Characterization of the finite simple groups by order and set of element orders

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Bath 2009

Shi's Conjecture

Wujie Shi, 1987

Search of arithmetical properties that determine a finite group uniquely.

Among arithmetical properties

- the order of the group $|G|$
- the set of element orders (the spectrum) $\omega(G)$

Problem: Is it true that every finite simple group is uniquely determined by its order and spectrum in class of finite groups?

More precisely

Question 12.39 in the Kourovka Notebook

Is it true that a finite group and a finite simple group are isomorphic if they have the same orders and sets of element orders?

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Answer: Yes

History

By 2003 the conjecture was verified for most of finite simple non-abelian groups.

- sporadic groups (Shi, 1987)
- alternating groups (Shi and Bi, 1992)
- Suzuki and Ree groups (Shi and Bi, 1991)
- exceptional groups of Lie type (Shi, 1994)
- linear groups (Shi and Bi, 1991)
- unitary groups (Shi and Cao, 2002)
- $D_n(q)$ with n odd and ${}^2D_n(q)$ (Shi and Xu, 2003)

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Remains: $B_n(q)$, $C_n(q)$ and $D_n(q)$ with n even

- $\omega(B_n(q)) \neq \omega(C_n(q))$ when q odd and $n > 2$ (Shi, 2007; G., 2007)

Main result

Theorem (Mazurov-Vasil'ev-G, 2009)

If L is one of $D_n(q)$ with n even, $B_n(q)$, $C_n(q)$ and G is a finite group such that $|G| = |L|$ and $\omega(G) = \omega(L)$ then $G \simeq L$.

Corollary

If L is a finite simple group and G is a finite group such that $|G| = |L|$ and $\omega(G) = \omega(L)$ then $G \simeq L$.

Composition structure of G

L is a finite non-abelian simple group
G is a finite group with $|G| = |L|$ and $\omega(G) = \omega(L)$
What can we say about composition factors of G?

Proposition (Vasil'ev-Vdovin, 2005)

If L is not alternating and differs from $A_2(3)$, ${}^2A_3(3)$, $C_2(3)$, then G has exactly one non-abelian composition factor.

$$S \simeq \mathrm{Inn}(S) \leq G/O_\infty(G) \leq \mathrm{Aut}(S)$$

$O_\infty(G)$ is the largest soluble normal subgroup of G, soluble radical
S is a finite non-abelian simple group

Remark

Proposition remains true if replace "$|G| = |L|$ and $\omega(G) = \omega(L)$" by "G and L have the same prime graphs".

Sketch of Shi's proof

L is a finite non-abelian simple group in characteristic p
G is a finite group with $|G| = |L|$ and $\omega(G) = \omega(L)$

$$S \leq G/O_\infty(G) \leq \mathrm{Aut}(S)$$

Principal task: prove that S is a group of Lie type in the same characteristic p
Idea: use Sylow p-subgroups

- Suppose p divides $|O_\infty(G)|$. Then exploiting primitive prime divisors show that $|O_\infty(G)|_p > |L|_p = |G|_p$.
- Derive $|S| < |S|_p^3$ and conlude that S is a group of Lie type in characterisric p.

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Why does this fail for $B_n(q)$ et cetera?

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Why does this fail for $B_n(q)$ et cetera?
"Bad" primitive prime divisors.

Sketch of our proof

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G is a finite group with $|G| = |L|$ and $\omega(G) = \omega(L)$

$$S \leq G/O_\infty(G) \leq \mathrm{Aut}(S)$$

Principal task: prove that S cannot be a group of Lie type in characteristic $v \neq p$
Idea: use Sylow v-subgroups

Fact 1

$|S|_v > u^{m^2/2}$, u is order of defining field and m is Lie rank of S
$|L|_v < (3q)^{3n/2}$, q is order of defining field and n is Lie rank of L

Fact 2 (Vasil'ev-Vdovin)

The rank of S cannot be much smaller than that of L, namely $m \geqslant n/2$.

$$u^{m^2/2} < |S|_v \leqslant |G|_v = |L|_v < (3q)^{3n/2} \leqslant (3q)^{3m}$$

$$u^m < (3q)^6 \quad (n \text{ disappears})$$

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Fact 3

$$\max \omega(S) \leqslant 2u^m$$

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Fact 4 (Vasil'ev-Vdovin)

A number of "large" element orders are inhereted by S, in particularly, $\max \omega(S) > q^{n/2}$.

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$$q^{n/2} < \max \omega(S) \leqslant 2u^m < 2(3q)^6$$

Contradiction for $n \geqslant 13$ and large q.
Actually, this works well for $n \geqslant 5$.

Alternating and sporadic factors

L is a finite simple symplectic or orthogonal group
G is a finite group with $\omega(G) = \omega(L)$ (not necessarily $|G| = |L|$)

$$S \leq G/O_\infty(G) \leq \mathrm{Aut}(S)$$

Theorem 1

S is neither alternating nor sporadic.

Groups of Lie type in the same characteristic

L is a finite simple symplectic or orthogonal group
G is a finite group with $\omega(G) = \omega(L)$ (not necessarily $|G| = |L|$)

$$S \leq G/O_\infty(G) \leq \mathrm{Aut}(S)$$

S is a group of Lie type in the same characteristic as L

Theorem 2

S is "close" to L:

- if $L = B_2(q)$ then $S \in \{B_2(q), A_1(q^2)\}$
- if $L \in \{B_3(q), C_3(q), D_4(q)\}$ then $S \in \{B_3(q), C_3(q), D_4(q), A_1(q^3), G_2(q)\}$
- if $L \in \{B_n(q), C_n(q), {}^2D_n(q)\}$ where $n \geqslant 4$ then $S \in \{B_n(q), C_n(q), {}^2D_n(q)\}$
- if $L = D_n(q)$ with n even then $S \in \{D_n(q), B_{n-1}(q), C_{n-1}(q)\}$
- if $L = D_n(q)$ with n odd then $S \simeq L$

Recognition by spectrum

L is a finite non-abelian simple group
G is a finite group with $\omega(G) = \omega(L)$ (isospectral to L)
Questions:

- Is it true that $G \simeq L$?
- How many are there finite groups isospectral to L?
- If finitely many, is it true that they are "close" to L?

Recognition problem

Given a finite simple group L, find the number $h(L)$ of finite groups isospectral to L. If $h(L)$ is finite (L is almost recognizable by spectrum), describe these groups.

Solution for linear groups in characteristic 2

$L = PSL_n(q)$ is a simple linear groups, $q = 2^\alpha$
$d = (n, q-1)$, $f = ((q-1)/d, \alpha)$, $f_d = \pi(d)$-part of f
φ is a field automorphism of L of order f_d
$\tau(f_d)$ = number of divisors of f_d

Shi, 1987; Mazurov-Xu-Cao, 2000; Zavarnitsine-Mazurov, 2007; Mazurov-Chen, 2008; Vasil'ev-G., 2008

Theorem

Suppose G is a finite group and $\omega(G) = \omega(L)$.

- If $n-1$ is a power of 2 then $G \simeq L$ ($h(L) = 1$).
- Otherwise, $G \simeq L_1$ where $L \leq L_1 \leq L\langle\varphi\rangle$ ($h(L) = \tau(f_d)$).

References

1. *M. Xu and W. Shi*, Pure quantitative characterization of finite simple groups $^2D_n(q)$ и $D_l(q)$ (l odd), Algebra Colloq., **10**, No. 3 (2003), 427–443.
2. *A.V. Vasil'ev*, On connection between the structure of a finite group and properties of its prime graph, Siberian Math. J., **46**, No. 3 (2005), 511–522.
3. *A.V. Vasiliev and E.P. Vdovin*, An adjacency criterion for the prime graph of a finite simple group, Algebra and Logic, **44**, No. 6 (2005), 381–406.

References

1. *M. Xu and W. Shi*, Pure quantitative characterization of finite simple groups ${}^2D_n(q)$ и $D_l(q)$ (l odd), Algebra Colloq., **10**, No. 3 (2003), 427–443.
2. *A.V. Vasil'ev*, On connection between the structure of a finite group and properties of its prime graph, Siberian Math. J., **46**, No. 3 (2005), 511–522.
3. *A.V. Vasiliev and E.P. Vdovin*, An adjacency criterion for the prime graph of a finite simple group, Algebra and Logic, **44**, No. 6 (2005), 381–406.
4. *M.A. Grechkoseeva*, Recognition by spectrum for finite simple linear groups over fields of characteristic 2, Algebra and Logic, **47**, No. 4 (2008), 229–241.
5. *A.V. Vasil'ev and M.A. Grechkoseeva*, Recognition by spectrum for finite simple linear groups of small dimensions over fields of characteristic 2, Algebra and Logic, **47**, No. 5 (2008), 314–320.