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ON A GRAPH RELATED TO MAXIMAL SUBGROUPS OF GROUPS

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I. **Introduction.**

Denote by G a finitely generated non-trivial group and let

$$\mathcal{M}(G) \;=\; \text{the class of maximal subgroups of } G.$$

We define

THE **MAXIMAL GRAPH** $\Gamma(G)$ OF G

as follows:

The **vertices** of $\Gamma(G)$ are

the maximal subgroups of G

and if $M_1, M_2 \in \mathcal{M}(G)$, with $M_1 \neq M_2$, then

(M_1, M_2) *is an* **edge** *of* $\Gamma(G)$ *if* $M_1 \cap M_2 \neq 1$.

This talk will concentrate on our results concerning finite groups.

II. **Finite groups**

Suppose that G is a finite group and consider the following three problems.

Problem 1.

When does $\Gamma(G)$ *have exactly one vertex?*

The question is:

When does G *have a unique maximal subgroup?*

It is easy to see that this happens if and only if

G *is a cyclic group of prime power order.*

Our next problem is

Problem 2.

When does $\Gamma(G)$ *have no edges?*

This problem is not as easy as Problem 1.

The question is:

When does G *have the property that* **any** *two distinct maximal subgroups of* G *intersect trivially?*

We showed that this happens if and only if one of the following statements holds:

(1) G is cyclic of prime power order (here there exists only one maximal subgroup);

(2) G is elementary abelian of order p^2;

(3) G is cyclic of order pq, where p, q are distinct primes,

(4) $G = P \rtimes Q$, where P is a normal elementary abelian p-group, $|Q| = q$, with q a prime distinct from p, and Q acts on P irreducibly and fixed point freely.

It is clear that in cases (2)-(4)

$\Gamma(G)$ *has at least two vertices*

and since it has no edges, it **is disconnected**.

This observation raises the third problem:

Problem 3.

When is $\Gamma(G)$ disconnected?

The surprising answer is:

Theorem 1. *Let G be a finite group. Then $\Gamma(G)$ is disconnected if and only if one of the above statements (2)-(4) holds.*

Or, in other words, $\Gamma(G)$ is disconnected if and only if

it has at least two vertices and no edges.

In order to deal with the proof of Theorem 1, assume first that G is a **non-simple** finite group.

Then there exists a proper minimal normal subgroup N of G and if $M \in \mathcal{M}(G)$, then

$$\textit{either } M \cap N \neq 1 \textit{ or } M \cap N = 1 \textit{ and } G = N \rtimes M.$$

This observation leads rather easily to the following conclusion:

Theorem 2. *Let G be a finite* **non-simple** *group. Then $\Gamma(G)$ is disconnected if and only if one of the above statements (2)-(4) holds.*

So it remains to investigate finite simple groups.

III. **Finite simple groups**

Concerning finite simple group, we proved the following basic theorem.

Theorem 3. *Let G be a finite simple group. Then:*

(1) $\Gamma(G)$ *is connected, and*
(2) $diam(\Gamma(G)) \leq 62$.

Our bound, 62, for the diameter of $\Gamma M(G)$ in the finite simple case, is most certainly not best possible.

Theorem 3 implies that groups with a disconnected maximal graph are non-simple. Hence Theorem 1 follows from Theorem 2.

In this talk we shall concentrate on the proof of the connectedness in Theorem 3.

In order to investigate the connectedness of $\Gamma(G)$ for finite simple groups, it is useful to consider another related graph, namely

THE **PRIME GRAPH** $\Pi(G)$ OF G.

This graph is defined as follows:

The **vertices** of $\Pi(G)$ are

the primes dividing the order of G

and if p, q are two such distinct primes, then

(p, q) *is an* **edge** *of* $\Pi(G)$
if G *contains an element of order* pq.

This graph was first investigated by K.Gruenberg and O.Kegel, and then by many authors, of which we shall mention only J.S.Williams (1981), M.S. Lucido (1999) and N.Chigira, N.Iiyori and H.Yamaki (2000).

The connection between $\Gamma(G)$ and $\Pi(G)$ is expressed in the following lemma. Denote by d the distance in the graph $\Gamma(G)$.

Lemma 4. *Let G be a finite group and let*

$$M, M_1 \in \mathcal{M}(G), \quad p \mid |M| \text{ and } q \mid |M_1|,$$

where p, q are distinct primes. Moreover, suppose that

$$p \text{ and } q \text{ are connected in } \Pi(G).$$

Then there exists $f \in G$ such that

$$M \text{ and } M_1^f \text{ are connected in } \Gamma(G)$$

and

$$d(M, M_1^f) \leq 20.$$

The bound $d(M, M_1^f) \leq 20$ of Lemma 4 was useful in the proof the inequality $diam(\Gamma(G)) \leq 62$ in Theorem 3.

Now we shall **sketch** the proof of Theorem 3. Here it is again

Theorem 3. *Let G be a finite simple group. Then:*

1. $\Gamma(G)$ *is connected, and*
2. $diam(\Gamma(G)) \leq 62$.

We shall concentrate on the connectedness.

If $M, M_1 \in \mathcal{M}(G)$, then $M \sim M_1$ will denote that

$$M \textit{ is connected to } M_1 \textit{ in } \Gamma(G)$$

and $M \nsim M_1$ will denote that they are *not connected* in $\Gamma(G)$.

We shall need the following proposition:

Proposition 5. *Let G be a finite non-abelian simple group and let $M, M_1 \in \mathcal{M}(G)$. Then the following statements hold:*

(1) *If $(|M|, |M_1|) > 1$, then $M \sim M_1^f$ for some $f \in G$.*
(2) *If $|M|$ and $|M_1|$ are even, then $M \sim M_1$.*
(3) *Suppose that $\Gamma(G)$ is non-connected. Then:*
*(**a**) $\exists M \in \mathcal{M}(G)$ such that $M \nsim M_e$ for* **any** *$M_e \in \mathcal{M}(G)$ of* **even** *order;*
*(**b**) if p is a prime, $p \mid |M|$ and $P \neq 1$ is a p-subgroup of G, then $N_G(P)$ is of* **odd order**.

So suppose that $\Gamma(G)$ is **non-connected** and we shall derive a contradiction.

1. By Prop. 5, $\exists M \in \mathcal{M}(G)$ of odd order such that $M \nsim M_e$ for any $M_e \in \mathcal{M}(G)$ of even order.

Let q be a **minimal** prime dividing the order of any $M_1 \in \mathcal{M}(G)$ such that $M_1 \nsim M_e$ for any $M_e \in \mathcal{M}(G)$ of even order. Such q exists by the previous statement.

Then, by Prop. 5,

if $M_1 \in \mathcal{M}(G)$ *and* $q \mid |M_1|$*, then* $M_1 \nsim M_e$ *for any* $M_e \in \mathcal{M}(G)$ *of even order.*

In particular, it follows by Lemma 4 that

q *is not connected to* 2 *in* $\Pi(G)$.

2. Let π be the connected component of $\Pi(G)$ containing q. By **1**,

$$2 \notin \pi$$

and hence by Williams(1981),

$\exists$ *a* **nilpotent** *Hall π-subgroup H of G*
such that

$$H \cap H^g = 1 \quad \forall g \in G - N_G(H), \quad \text{and}$$
$$C_G(h) \leq H \quad \forall h \in H - \{1\}.$$

If $N_G(H) = H$, then G is a Frobenius group with H as a Frobenius complement and G is **non-simple**, a contradiction.

Hence $N_G(H) > H$ and since H is a Hall subgroup of G,

$\exists y \in N_G(H) \backslash H$ *of prime order p, where $p \notin \pi$.*

In particular, $y \notin H$ and $H\langle y \rangle$ is a proper subgroup of G.

3. Let $H\langle y\rangle \leq M \in \mathcal{M}(G)$.

As $q \mid |M|$, it follows by **1** that

$M \not\sim M_e$ for any $M_e \in \mathcal{M}(G)$ of even order.

But also $p \mid |M|$, so it follows from the minimality of q that

$$p > q.$$

Moreover, we can prove, like in Prop.5(3), that if r is any prime dividing $|M|$ and

$$R \neq 1 \textit{ is an } r\textit{-subgroup of } G,$$

then $N_G(R)$ is of **odd** order.

Finally,

4. Let $Q \leq H$ be a Sylow q-subgroup of H and hence also of G.

Then $y \in N_G(Q)$, since H is nilpotent and $y \in N_G(H)$.

Since q is not connected to 2 in $\Pi(G)$ (see **1**), it follows by Chigira, Iiyori and Yamaki(2000) that

$$Q \textit{ is abelian.}$$

Now Q is an abelian Sylow q-subgroup of the simple group G and we may apply a recent result of Sawabe(2007).

He proved that Q must satisfy one of the following conditions:

(i) $N_G(Q)/C_G(Q)$ contains an involution;

(ii) Q is cyclic;

(iii) $G \cong PSL(2, q^e)$ for some $q^e > 3$.

We shall derive a contradiction by showing that none of the Sawabe conditions is satisfied in our situation.

First, it is quite easy to show that $\Gamma(PSL(2, q^e))$ is connected for $q^e > 3$, so (iii) is impossible.

Second, it follows by **3**, that $N_G(Q)$ is of odd order, so (i) is impossible.

Finally, suppose that (ii) holds and Q is a **cyclic** group of order, say, q^e. Thus

$$[N_G(Q) : C_G(Q)] \mid q^{e-1}(q-1).$$

But $y \in N_G(Q)$ and $|y| = p$, a prime.

Moreover, $p > q$ by **3**, and hence $p \nmid q^{e-1}(q-1)$.

Consequently, $y \in C_G(Q)$.

But $1 < Q \leq H$ and in view of Williams' result

$$C_G(h) \leq H \quad \forall h \in H - \{1\}$$

it follows from $y \in C_G(Q)$ that $y \in H$, a contradiction.

This final contradiction implies that $\Gamma(G)$ **is connected**.

THE END