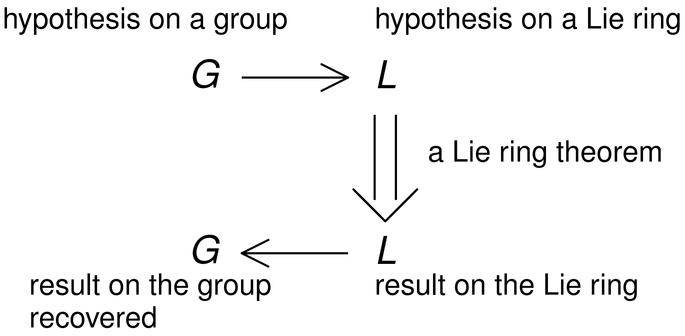
Lie Ring Method Applied to a Frobenius group of automorphisms

Evgeny KHUKHRO

Lie ring (algebra) method:



1. A hypothesis on a group is translated into a hypothesis on a Lie ring constructed from the group in some way.
2. Then a theorem on Lie rings is proved (or used).
3. Finally, a result about the group must be recovered from the Lie ring information obtained.

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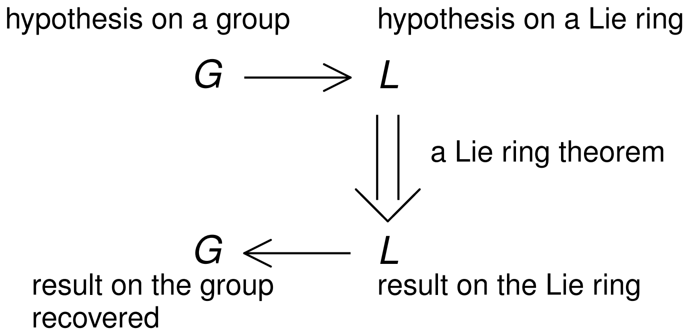

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Lie Ring Method for Arbitrary Groups

including finite groups (where, e.g., Baker–Campbell–Hausdorff formula cannot be applied):

Associated Lie Ring:

For any group G: $L(G) = \bigoplus_i \gamma_i(G)/\gamma_{i+1}(G)$

Lie product for homogeneous elements:
$[a + \gamma_{i+1},\ b + \gamma_{j+1}]_L \ = \ [a, b]_G + \gamma_{i+j+1}$

extended to the direct sum by linearity.

Pluses: Always exists.

Nilp. class of G = Nilp. class of $L(G)$

Minuses: Only about $G/\bigcap \gamma_i(G)$,
so only for (residually) nilpotent groups.
Even for these, some information may be lost:
e. g., derived length may become smaller.

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Idea:

How results about Lie rings,
even rather “extravagant”,
find applications in Group Theory.

$(\mathbb{Z}/n\mathbb{Z})$-graded Lie ring with $L_0 = 0$

$L = L_0 \oplus L_1 \oplus \cdots \oplus L_{n-1}$

L_i additive subgroups satisfying $[L_i, L_j] \subseteq L_{i+j \pmod n}$.

Higman, Kostrikin, Kreknin:

if $L_0 = 0$, then L is soluble of derived length $\leqslant k(n)$;

if in addition $n = p$ is a prime, then nilpotent of class $\leqslant h(p)$.

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Fixed-point-free automorphisms

Hence the same results for a Lie ring M with a fixed-point-free automorphism φ of order n: after adjoining a primitive nth root of unity ω we obtain $M = M_0 + M_1 + \cdots + M_{n-1}$ for $M_i = \{x \in M \mid \varphi(x) = \omega^i x\}$, where $[M_i, M_j] \subseteq M_{i+j \,(\mathrm{mod}\, n)}$ and $M_0 = 0$ (the fact that the sum is not direct in general is inessential).

Corollary (Higman):

If a (locally) nilpotent group G has a fixed-point-free automorphism $\varphi \in Aut\, G$ of prime order p, $C_G(\varphi) = 1$, then G is nilpotent of class $\leqslant h(p)$.

Proof: consider $M = L(G)$ with the induced automorphism....

Also true for any finite group G, nilpotent by Thompson.

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Open problem:

Does an analogue of Kreknin's theorem hold for nilpotent groups with a fixed-point-free automorphism of arbitrary finite order n? i. e. is derived length $\leqslant f(n)$?

(For arbitrary finite groups with a fixed-point-free automorphism everything already reduced to nilpotent groups by classification and Hall–Higman–type theorems.)

Here $L = L(G)$ does not work as derived length is not preserved.

"Modular situation" a success!

Nevertheless, Kreknin's theorem was successfully applied to finite p-groups with an automorphism of order p^k and to pro-p-groups of given coclass in the papers of Jaikin-Zapirain, Khukhro, Medvedev, Shalev, Shalev–Zel'manov.

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Makarenko and Khukhro:

then L contains a soluble (for n prime, nilpotent) ideal of n-bounded derived length (nilpotency class) and of (n, r)-bounded codimension.

Immediate corollaries for Lie rings with automorphism $\varphi^n = 1$ and $\dim C_L(\varphi) = r$ (or $|C_L(\varphi)| = r$).

Corollary for nilpotent groups with an "almost fixed-point-free" automorphism of prime order p.

Most recent corollary for nilpotent groups with an automorphism of prime order p that is "almost fixed-point-free" in the sense of rank of $C_G(\varphi)$.

Difficult recovery $G \longleftarrow L$, as there is no good correspondence for subgroups$\leftrightarrow$subrings.

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$(\mathbb{Z}/n\mathbb{Z})$-graded Lie ring L with few non-zero components

Suppose that in $L = L_0 \oplus \cdots \oplus L_{n-1}$ there are only d nonzero components among the L_i.

Shalev and Khukhro:

if in addition $L_0 = 0$, then L is soluble (for n prime, nilpotent) of d-bounded derived length (class).

Applied to groups of bounded rank with almost fixed-point-free automorphisms.

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$(\mathbb{Z}/n\mathbb{Z})$-graded Lie ring L with many commuting components

Let L be a $(\mathbb{Z}/n\mathbb{Z})$-graded Lie ring.

Theorem 1

Suppose that some m for we have

$$|\{i \mid [L_k, L_i] \neq 0\}| \leqslant m,$$

i. e. each component L_k commutes with all but at most m components. If $L_0 = 0$, then L is soluble (for n prime, nilpotent) of m-bounded derived length (class).

2-Frobenius groups

Theorem 2

Let BC be a finite Frobenius group with complement C of order $|C| = t$ acting on a finite group A so that AB is also a Frobenius group, and let $(t, |A|) = 1$. If $C_A(C)$ is abelian, then A is nilpotent of t-bounded class.

A is nilpotent by Thompson, of nilpotency class bounded in terms of the least prime divisor of $|B|$ by Higman. Theorem 2 gives a better bound if $|C|$ has a small divisor.

2-Frobenius groups arise, in particular, in the study of the Gruenberg–Kegel prime graphs of finite groups. Earlier the result of Theorem 2 was proved by Mazurov for $t = 2$ and 3.

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Let L be a $(\mathbb{Z}/n\mathbb{Z})$-graded Lie ring.

Theorem 3

Suppose that for some m for $k \neq 0$ we have

$$|\{i \mid [L_k, L_i] \neq 0\}| \leqslant m,$$

i. e. each component L_k for $k \neq 0$ commutes with all but at most m components. If $\dim L_0 = r$ (or $|L_0| = r$), then L contains a soluble (for n prime, nilpotent) ideal of m-bounded derived length (class) and of (n, r)-bounded codimension.

Theorem 4

Let BC be a Frobenius group with kernel B of prime order p and with complement C of order t acting on a finite group A so that $|C_A(B)| = s$ and $(|A|, |BC|) = 1$. If $C_A(C)$ is abelian, then A contains a characteristic subgroup of (p, s)-bounded index that is nilpotent of t-bounded class.

By Fong, Hartley–Meixner, Khukhro, A has a subgroup of (p, s)-bounded index that is nilpotent of p-bounded class, and it can be chosen characteristic by Makarenko–Khukhro. Theorem 3 gives a better bound for the class if t has a small divisor.

Frobenius groups of automorphisms

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