Torsion Units in Integral Group Rings of Sporadic Simple Groups

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(joint work with Victor Bovdi, Eric Jespers, Steve Linton, Salvatore Siciliano et al.)

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Introduction

Notation

G – finite group

$\mathbb{Z}G$ – its integral group ring

$U(\mathbb{Z}G)$ – unit group of $\mathbb{Z}G$

$V(\mathbb{Z}G)$ – subgroup of normalized units

Conjecture ZC-1 (Zassenhaus, 1974)

Every torsion unit $u \in V(\mathbb{Z}G)$ is conjugate within the rational group algebra $\mathbb{Q}G$ (that is, *rationally conjugated*) to an element of G

Prime graph of a group

A question by Wolfgang Kimmerle relates ZC-1 with the *prime graph* of a group and may be regarded as weak version of ZC-1.

Definition

$\pi(G)$ – *prime graph* of a group G:

- vertices of $\pi(G)$ correspond to primes dividing orders of torsion elements of the group $|G|$
- an edge connects p and q $\Leftrightarrow$ G has an element of order pq

Problem (Oberwolfach Reports Vol.4, no.4, Report No.55/2007)

(Problem 21) *For a finite group G, is it true that* $\pi(V(\mathbb{Z}G)) = \pi(G)$*?*

W. Kimmerle and C. Höfert (2005) gave positive answer for finite Frobenius and solvable groups, saying that "*for non-soluble groups the known methods admit so far only results for few cases*"

Research program and technique

Determining $\pi(V(\mathbb{Z}G))$ for sporadic simple groups

- Motivated by this question, jointly with Victor Bovdi we started the project to determine $\pi(V(\mathbb{Z}G))$ for sporadic simple groups
- Considering it the ZC-1 context, not restricting our attention only to units of order pq
- Using the method by I. S. Luthar and I. B. S. Passi (1989) with recent extensions by M. Hertweck (2006) (let me suggest an acronym HeLP for this method):
  - Produces constraints imposed on coefficients of torsion units in $V(\mathbb{Z}G)$ and (in the ideal situation) eliminates all unwanted cases
  - First was used to prove ZC-1 for A_5 (Luthar & Passi, 1989) and S_5 (Luthar & Trama, 1991)
  - Extended on Brauer characters by M. Hertweck in 2006
  - Advantage: uses ordinary and Brauer character tables without explicit computations in group rings and may be implemented in the computational algebra system GAP (so we may call it F1)

Partial augmentations

Notation

$u = \sum \alpha_g g$ – normalized torsion unit of $\mathbb{Z}G$

$C_1, \dots, C_n$ – conjugacy classes of G

$\nu_i = \nu_i(u) = \varepsilon_{C_i}(u)$ – partial augmentation of u, corresponding to the conjugacy class C_i :

$$\nu_i(u) = \sum_{g \in C_i} \alpha_g$$

Finiteness of the search space

Main criterion for ZC-1

A torsion unit of $V(\mathbb{Z}G)$ is rationally conjugated to an element of G, if all but one partial augmentations are equal to zero (Marciniak–Ritter–Sehgal–Weiss, 1987; Luthar–Passi, 1989)

Partial augmentations of torsion units are bounded

$\nu_i(u)^2 \leq |C_i|$ and $\sum_{i=1}^{n} \frac{\nu_i(u)^2}{|C_i|} \leq 1$ (Hales–Luthar–Passi, 1990)

Orders of torsion units are bounded

The order of a torsion unit of $V(\mathbb{Z}G)$ divides the exponent of G (Cohn–Livingstone, 1965)

Further reductions

From the Berman–Higman Theorem (1955) $\mathrm{tr}(u) = \nu_1 = 0$

Theorem (Marciniak–Ritter–Sehgal–Weiss, 1987; Luthar–Passi, 1989)

If p is a prime dividing the order of a representative of C but not the order of u then $\varepsilon_C(u) = 0$

Example

If $G = M_{11}$, for units of order 2 we have:

$$\nu_1 = \nu_{3a} = \nu_{5a} = \nu_{6a} = \nu_{11a} = \nu_{11b} = 0$$

$$\nu_{2a} + \nu_{4a} + \nu_{8a} + \nu_{8b} = 1$$

Improved restrictions

Theorem (Hertweck, 2005–2006; generalizes MRSW-LP result)

Let p be a prime such that the order of the representative of C is $p^n s$, and the order of u is $p^m t$. Then $n > m \;\Rightarrow\; \varepsilon_C(u) = 0$

Example for $G = M_{11}$

- for a unit of order 2 $\nu_{4a} = \nu_{8a} = \nu_{8b} = 0$, so $\nu_{2a} = 1$ is the only one non-zero partial augmentation, and ZC-1 holds for order 2
- for a unit of order 10 $\nu_{4a} = \nu_{8a} = \nu_{8b} = 0$, so $\nu_{2a} + \nu_{5a} = 1$

The main component of the method

Theorem (Luthar–Passi, 1989; modular case - Hertweck, 2005)

- *p is either 0 or a prime divisor of $|G|$*
- *$u \in V(\mathbb{Z}G)$ is a normalized torsion unit of order k*
- *if $p \neq 0$, then k and p must be coprime*
- *z is a complex primitive k-th root of unity*
- *χ is a classical character or a p-Brauer character of G*
- *for all l the number*

$$\mu_l(u,\chi,p) = \frac{1}{k} \sum_{d|k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}} \left\{ \chi(u^d) z^{-dl} \right\}$$

is a non-negative integer which is not greater than $\deg(\chi)$

How to compute $\mu_l(u, \chi, p)$?

$$\frac{1}{k}\sum_{d|k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}}\left\{\chi(u^d)z^{-dl}\right\} = \frac{1}{k}\Big(Tr_{\mathbb{Q}(z)/\mathbb{Q}}\{\chi(u)z^{-l}\} + Tr_{\mathbb{Q}(z^{d_1})/\mathbb{Q}}\{\chi(u^{d_1})z^{-d_1 l}\} + \dots + Tr_{\mathbb{Q}(z^{d_i})/\mathbb{Q}}\{\chi(u^{d_i})z^{-d_i l}\} + \dots + Tr_{\mathbb{Q}(z^{d_{s-1}})/\mathbb{Q}}\{\chi(u^{d_{s-1}})z^{-d_{s-1} l}\} + \chi(1)\Big)$$

where $d_i|k$, $d_i > 1$ and the summand $\chi(1)$ comes from $d_s = k$.
For any character χ we have that

$$\chi(u) = \sum_{j=1}^{n} \chi(h_j)\,\nu_j\,, \qquad \chi(u^{d_i}) = \sum_{j=1}^{n} \chi(h_j)\,\nu_j^{(k_i)},$$

where h_j is a representative of the conjugacy class C_j, and $\nu_j^{(k_i)}$ is the partial augmentation for the conjugacy class C_j for an element u^{d_i} of order $k_i = d/d_i$.

Computation of $\mu_l(u, \chi, p)$ continued

Since the trace is a linear mapping, this gives us $\mu_l(u, \chi, p)$ as a linear combination of corresponding partial augmentations:

$$
\begin{aligned}
\mu_l(u, \chi, p) = & \alpha_1 \nu_1 + \cdots + \alpha_n \nu_n + \\
& \alpha_1^{(k_1)} \nu_1^{(k_1)} + \cdots + \alpha_n^{(k_1)} \nu_n^{(k_1)} \ + \ \cdots + \\
& \alpha_1^{(k_i)} \nu_1^{(k_i)} + \cdots + \alpha_n^{(k_i)} \nu_n^{(k_i)} \ + \ \cdots + \\
& \alpha_1^{(k_{s-1})} \nu_1^{(k_{s-1})} + \cdots + \alpha_n^{(k_{s-1})} \nu_n^{(k_{s-1})} \ + \ \chi(1) \geq 0
\end{aligned}
$$

Since all trace values must exist, in addition to already eliminated partial augmentations we might obtain that some more of them are zero to avoid contradiction when character value does not belong to the appropriate field.

Constraint satisfaction problem

Now to form the CSP (constraint satisfaction problem) we take:

- all inequalities for $\mu_l(u, \chi, p_i)$ for normalized units of order k for all $0 \leq l < k$, χ and p_i
- similarly produced on earlier steps systems of inequalities with indeterminates $\nu_1^{(k_i)}, \dots, \nu_n^{(k_i)}$ for normalized units of order k_i
- equation $\nu_1 + \cdots + \nu_n = 1$
- equations $\nu_1^{(k_i)} + \cdots + \nu_{n_i}^{(k_i)} = 1$ for every order k_i

Example

- Elements of order 119 in the Held sporadic simple group *He*
- Conjugacy classes C_{7a}, C_{7b}, C_{7c}, C_{7d}, C_{7e} and C_{17a}, C_{17b}
- The general form of the constraint for a character χ will be

$$\frac{1}{119}\Big(\alpha_1\nu_{7a}+\alpha_2\nu_{7b}+\alpha_3\nu_{7c}+\alpha_4\nu_{7d}+\alpha_5\nu_{7e}+\alpha_6\nu_{17a}+\alpha_7\nu_{17b}+ \alpha_1^{(17)}\nu_{7a}^{(17)}+\alpha_2^{(17)}\nu_{7b}^{(17)}+\alpha_3^{(17)}\nu_{7c}^{(17)}+\alpha_4^{(17)}\nu_{7d}^{(17)}+\alpha_5^{(17)}\nu_{7e}^{(17)}+ \alpha_1^{(7)}\nu_{17a}^{(7)}+\alpha_2^{(7)}\nu_{17b}^{(7)}+\chi(1)\Big)\geq 0,$$

- We also add earlier calculated systems for elements of orders 7 and 17 with respect to indeterminates $\nu_{17x}^{(7)}$ and $\nu_{7x}^{(17)}$
- We add three conditions that the sum of partial augmentations is equal to one for normalized units of orders 7, 17, 119
- Alternatively, solving first all systems for orders 7 and 17, we can substitute all possible combinations and then solve 234842 smaller systems for elements of order 119, or substitute either 7 or 17 but not both

Example: *Co*1

- Such "unrolling" optimizes elimination of order 77 for *Co*1
- Units of order 11 are rationally conjugate to group elements, but for order 7 standard HeLP yields 2 trivial and 45 non-trivial cases
- Instead of enumerating them and solving 47 systems, we get one:

$$\nu_{7a} + \nu_{7b} + \nu_{11a} = 1; \qquad \nu_{11a}^{(7)} = 1; \qquad \nu_{7a}^{(11)} + \nu_{7b}^{(11)} = 1$$

$$\mu_{11}(u, \chi_2, 0) = \tfrac{1}{77}(-100\nu_{7a} - 30\nu_{7b} - 10\nu_{11a} + 10\nu_{11a}^{(7)} - 10\nu_{7a}^{(11)} - 3\nu_{7b}^{(11)} + 276) \geq 0;$$

$$\mu_{0}(u, \chi_3, 0) = \tfrac{1}{77}(300\nu_{7a} + 300\nu_{7b} + 120\nu_{11a} + 20\nu_{11a}^{(7)} + 30\nu_{7a}^{(11)} + 30\nu_{7b}^{(11)} + 299) \geq 0;$$

$$\mu_{0}(u, \chi_4, 0) = \tfrac{1}{77}(840\nu_{7a} + 84\nu_{7a}^{(11)} + 1771) \geq 0;$$

$$\mu_{1}(u, \chi_4, 0) = \tfrac{1}{77}(14\nu_{7a} - 14\nu_{7a}^{(11)} + 1771) \geq 0;$$

$$\mu_{7}(u, \chi_4, 0) = \tfrac{1}{77}(-84\nu_{7a} + 84\nu_{7a}^{(11)} + 1771) \geq 0;$$

$$\mu_{11}(u, \chi_4, 0) = \tfrac{1}{77}(-140\nu_{7a} - 14\nu_{7a}^{(11)} + 1771) \geq 0;$$

$$\mu_{0}(u, \chi_7, 0) = \tfrac{1}{77}(840\nu_{7a} - 120\nu_{11a} - 20\nu_{11a}^{(7)} + 84\nu_{7a}^{(11)} + 27300) \geq 0;$$

$$\mu_{0}(u, \chi_{15}, 13) = \tfrac{1}{77}(-420\nu_{7a} + 60\nu_{11a} + 10\nu_{11a}^{(7)} - 42\nu_{7a}^{(11)} + 474145) \geq 0;$$

This system has no solutions, covers all 47 cases and, moreover, does not require preliminary dealing with systems for order 7.

Grouping partial augmentations together

First, note that HeLP method does not require to use only irreducible characters. Now, let

- u – normalized unit of order pq
- p and q – primes such that G has no element of order pq
- ν_k is the sum of partial augmentations of u with respect to all conjugacy classes of elements of order k, e.g. $\nu_2 = \nu_{2a} + \nu_{2b}$.
- Then $\nu_p + \nu_q = 1$ and $\nu_k = 0$ for $k \notin \{p, q\}$.
- We will say that χ (an ordinary character or a Brauer character in characteristic not dividing pq) is a *(p, q)-constant character*, if it constant on all elements of order p and constant on all elements of order q. Then we have

$$\chi(u) = \nu_p \chi(C_p) + \nu_q \chi(C_q),$$

where $\chi(C_t)$ is the value of χ on any element of order t from G

Rewriting $\mu_l(u, \chi)$

For normalized units of order *pq* we have that

$$\mu_l(u,\chi) = \frac{1}{pq}\Big(\chi(1) + Tr_{\mathbb{Q}(z^p)/\mathbb{Q}}\{\chi(u^p)z^{-pl}\} + Tr_{\mathbb{Q}(z^q)/\mathbb{Q}}\{\chi(u^q)z^{-ql}\} + Tr_{\mathbb{Q}(z)/\mathbb{Q}}\{\chi(u)z^{-l}\} \Big)$$

are nonnegative integers, and if χ is (p,q)-constant character then

$$\mu_l(u,\chi) = \frac{1}{pq}\left(m_1 + \nu_p m_p + \nu_q m_q\right),$$

where

$$m_1 = \chi(1) + \chi(C_q)\,Tr_{\mathbb{Q}(z^p)/\mathbb{Q}}(z^{-pl}) + \chi(C_p)\,Tr_{\mathbb{Q}(z^q)/\mathbb{Q}}(z^{-ql}),$$
$$m_p = \chi(C_p)\,Tr_{\mathbb{Q}(z)/\mathbb{Q}}(z^{-l}), \qquad m_q = \chi(C_q)\,Tr_{\mathbb{Q}(z)/\mathbb{Q}}(z^{-l}).$$

How much we can get from this?

- We are interested in a systematic search for (p, q)-constant characters that are capable of producing new constraints on partial augmentations
- If we have two (p, q)-constant characters χ_1 and χ_2, then $\chi_1 + \chi_2$ can not give us any further restrictions on partial augmentations
- The task is to find all (p, q)-constant characters that can not be represented as a sum of other (p, q)-constant characters. We will call such characters *(p, q)-irreducible characters*
- This can be performed by analyzing relative differences between values of irreducible characters on all conjugacy classes of the given order
- Discovering a “good character” reduces the number of cases to enumerate and allows to dramatically simplify the proof
- However, it might be tricky or even impossible to find it

Open questions

- O'Nan group: Using the 5-Brauer character $\chi = \chi_1 + \chi_3 + \chi_9 + \chi_{10}$ such that $\chi(C_3) = 26$ and $\chi(C_7) = 0$ with $l \in \{0, 1, 7\}$ we obtain the system of constraints that eliminates torsion units of order 21.
- However, for torsion units of order 33 one case still remains for *ON*: $(\nu_{3a}, \nu_{11a}) = (12, -11)$. Since we have one class of elements of order 3 and one of order 11, then every irreducible character is already (3,11)-irreducible, so in this case (p, q)-constant extension of the HeLP will not give us any new information.
- Besides that, for torsion units of order 57 for *ON* the best restriction we discovered so far using (p, q)-constant characters is $\nu_{3a} = -18$ and $\nu_{19a} + \nu_{19b} + \nu_{19c} = 19$
- 3^{rd} and 2^{nd} Conway groups: for order 35, we enumerated *all* (5,7)-constant characters and they can not eliminate two cases: $(\nu_{5a}, \nu_{5b}, \nu_{7a}) \in \{\ (3, 12, -14),\ (4, 11, -14)\ \}$,

Current status of the determination of $\pi(V(\mathbb{Z}G))$

Answered question of Kimmerle for 13 sporadic simple groups:

- Mathieu simple groups: M_{11}, M_{12} (jointly with S. Siciliano), M_{22} (jointly with S. Linton), M_{23}, M_{24}
- Janko simple groups (jointly with E. Jespers): J_1, J_2, J_3
- Higman-Sims simple group HS
- McLaughlin simple group McL
- Held simple group He (jointly with A. Grishkov)
- Rudvalis simple group Ru
- Suzuki simple group Suz (jointly with E.N. Marcos)

Obtained detailed descriptions:

- O'Nan simple group ON (jointly with A. Grishkov)
- Conway groups $Co3$ and $Co2$ (jointly with S. Linton)

Technical details

- We used computers to build and analyse constraints imposed on partial augmentations
- Implemented in GAP, at some stage will become available as a part of the LAGUNA package
- We use several solvers:
  - ECLiPSe (http://www.eclipse-clp.org, using development version of the GAP package IF)
  - MINION (http://minion.sourceforge.net, using GAP interface by Steve Linton) - much faster than ECLiPSE
  - Own solver implemented in GAP - shaped for our specific needs

Average total runtime spend in solvers, ms (excluding generation of CP solvers input):

Group	Number of systems	GAP solver	MINION solver	ECLiPSe solver
J_1	39	114.6	6.4	246.2
J_2	22	61.2	4.2	39.4
J_3	384	2251.2	62.8	956

on 8-core Dell Poweredge 2950 (`ardbeg`: 2 quad-core Intel Xeon 2.66 GHz / RAM 16 GB / CentOS Linux 4.5)