Engel conditions on orderable groups and in combinatorial problems

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Engel elements

G a group, $x, y \in G$, n a non-negative integer.
The commutator $[x, {}_n y]$ is defined, by induction, by the rules:

$$[x, {}_0 y] = x, \quad [x, {}_{n+1} y] = [[x, {}_n y], y].$$

$x \in G$ is a *right Engel element of G* (a *left Engel element of G*) if for each $g \in G$ there is an integer $n = n(x, g) \geq 0$ such that

$$[x, {}_n g] = 1 \ \ ([g, {}_n x] = 1).$$

If n can be chosen independently on g we say that x is a *right n-Engel element* (a *left n-Engel element*).
If every element of a group G is a right Engel element (equivalently every element of G is a left Engel element), then G is called an *Engel group*.
G is called an *n-Engel group* if

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- *G a nilpotent group of class c* $\Longrightarrow$ *G a c-Engel group.*
- *There exists an infinite 3-Engel group with trivial center, thus k-Engel groups need not to be nilpotent.*

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- *G a finite k-Engel group* $\Longrightarrow$ *G nilpotent [M. Zorn, 1937]*
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$(G, \leq)$ is a *partially ordered group* if, for any $x, y, a, b \in G$,

$$x \leq y \Longrightarrow axb \leq ayb.$$

If $(G, \leq)$ is a partially ordered group and the order $\leq$ is a total order in G, we say that $(G, \leq)$ is a *totally ordered group* or simply *an ordered group*.
G is an *orderable group* (an *O-group*) if there exists a total order $\leq$ such that $(G, \leq)$ is an ordered group.

Example

Any nilpotent torsion-free group is an orderable group.

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Theorem (A)

[Y. K. Kim, A. H. Rhemtulla, 1995] An orderable k-Engel group is nilpotent of class $\leq f(k)$.

It is very easy to see that an orderable group is always torsion-free and, as noticed before, every nilpotent torsion-free group is an orderable group, thus we could ask:

Question (A. I. Kokorin, problem 2.24 of The Kourovka Notebook)

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Lemma (1)

Let G be a k-Engel group, then the subgroup $< x >^{<y>}$ can be generated by k elements, for any $x, y \in G$.

Lemma (2)

Let G be a finitely generated k-Engel group. If H is normal in G and G/H is cyclic then H is finitely generated.

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A subgroup C of an ordered group is *convex* if $x \in C$, whenever $1 \leq x \leq c$ with $c \in C$.

A *relatively convex* subgroup of an O-group G is a subgroup convex under some order on G.

The quotient G/N of an O-group G is an O-group if and only if N is relatively convex.

If C, D are convex subgroups of an ordered group G, $C < D$ and there is not a convex subgroup H of G such hat $C < H < D$, we say that $C \mapsto D$ is a *convex jump in* G.

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Lemma (3)

A convex subgroup of an ordered k-Engel group $(G, \leq)$ is normal in G.

Proof.

- Let C be a convex subgroup of G, $g \in G$.
- The subgroup $g^{-1}Cg$ is also convex. For: $1 \leq a \leq g^{-1}bg, b \in C \Longrightarrow 1 \leq gag^{-1} \leq b \in C \Longrightarrow gag^{-1} \in C \Longrightarrow a \in g^{-1}Cg$.
- Either $g^{-1}Cg \subseteq C$ or $C \subseteq g^{-1}Cg$. Assume w.l.o.g. $C \subseteq g^{-1}Cg$. Then $C \subseteq g^{-i}Cg^{i}$, for any $i > 0$ and $g^{-i}Cg^{i} \subseteq C$, for any $i < 0$.
- Assume $C \subset g^{-1}Cg$ and let $c \in C$ such that $g^{-1}cg \notin C$.
- By Lemma 1, $< c >^{<g>} \subseteq g^{-s}Cg^{s}$, for some $s > 0$.
- Therefore $g^{-(s+1)}cg^{s+1} \in g^{-s}Cg^{s}$, from which $g^{-1}cg \in C$, a contradiction.

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Obviously an orderable group is an l-group. In 1988 N. Ya Medvedev proved the following

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Every k-Engel l-group is residually orderable.

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If G is an RO k-Engel group, in order to show that G is nilpotent it would be sufficient to prove that G is locally indicable, i.e. every non trivial finitely generated subgroup of G has an infinite cyclic factor group.

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A right partial order on the group G is said to be *archimedean* if, for any $a, b \in G, a > 1, b > 1$, there exists a positive integer n such that $b < a^n$.

By a result of Hölder an order on G is archimedean if and only if G is order-isomorphic to a subgroup of the additive group of the real numbers under the natural order.

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Theorem (P.L, M. Maj, A. H. Rhemtulla)

If an RO-group $(G, \leq)$ has no non-abelian free subsemigroups, then $\leq$ is a Conrad order.

Therefore we get the following

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Is it true that every RO k-Engel has no free non-abelian subsemigroups?

This is an old question for general k-Engel groups (see The Kourovka Notebook, Problem 2.82)
P. L. and M. Maj proved in 1997 that a right orderable 4-Engel group satisfies a non-trivial semigroup identity, hence it is nilpotent of bounded class. More generally G. Traustason proved in 1999 that any 4-Engel group satisfies a non-trivial semigroup identity and in 2005 G. Havas and M. Vaughan-Lee proved that 4- Engel groups are locally nilpotent. They are also Fitting groups by a result of G. Traustason.

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Notice that if we only assume that the ordered group G is an Engel group it is not true that G is necessarily nilpotent, as the following example shows.

Example

Let A be an associative algebra over a field K. An element $a \in A$ is called *nilpotent* if $a^n = 0$ for some positive integer depending on a. If all elements of A are nilpotent the A is called a $nil - algebra$. The algebra A is called *nilpotent* if there exists a positive integer n such that $a_1 a_2 \cdots a_n = 0$, for any $a_1, \cdots, a_n \in A$. Obviously every nilpotent algebra is a nil-algebra, the converse is not true. Let A be an associative algebra with a unit element 1 and B a nil-subalgebra of A. The elements of the form $1 + u$, $u \in B$, with the product of A form a group $G(B)$. It is easy to prove that this group is nilpotent if B is nilpotent.

E. S. Golod constructed in 1966, for any field K and any integer $d \geq 3$, a non-nilpotent d-generated associative algebra F such that every $(d-1)$-generated subalgebra is nilpotent. The group $G(F)$ is a non-nilpotent Engel group. V. V. Bludov, A.M.W. Glass and A. H. Rhemtulla noticed in 2005 that if K is of characteristic 0, then the group $G(F)$ is residually-(torsion-free nilpotent), thus it is also orderable.

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It is still open the following

Question (V. V. Bludov, problem 16.15 in The Kourovka Notebook)

Does the set of left Engel elements of an ordered group form a subgroup?

It is also also open the following

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Is every lattice ordered Engel group residually orderable?

Recently V. V. Bludov, A.M.W. Glass and A. H. Rhemtulla proved in 2005 the following interesting results

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If an orderable group is generated by left Engel elements, then every convex jump is central.

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If an orderable group G is an Engel group, then every two-generated subgroup of G has all convex jumps central.

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Engel conditions in combinatorial problems

Let X be a class of groups. Given a group G, let $\Gamma_{X^\circ}(G)$ be the simple graph whose vertices are the elements of G, and different vertices x and y are connected by an edge if the subgroup $\langle x, y\rangle$ belongs to the class X. The group G is said to be an $X^\circ - group$ if the graph $\Gamma_{X^\circ}(G)$ has no infinite totally disconnected subgraphs, i.e. in any infinite subset of G there exist different elements x, y such that $\langle x, y\rangle \in X$.
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For example this is true if $X = A$, where A denotes the class of all abelian groups: here $S(A) = Z(G)$; conversely B. H. Neumann, answering to a question posed by P. Erdös, proved he following result

Theorem (B. H. Neumann, 1976)

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Theorem (J.C. Lennox and J. Wiegold, 1981)

Let G be a finitely generated soluble group. Then G is an N°-group if and only if it is finite-by-nilpotent.

For any prime $p \geq 5$, M.R. Vaughan-Lee and J. Wiegold constructed in 1981 a countable locally finite group of exponent p which is perfect, and such that each of its 2-generator subgroups is nilpotent of bounded class. Hence the result of the previous theorem does not hold in general, even if we assume there is a bound for the nilpotence class of all 2-generated subgroups.

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For the class N_k°, where N_k is the class of nilpotent groups of class at most k, generalizing previous results by A. Abdollahi and B. Taeri and by C. Delizia, C. Delizia, A. H. Rhemtulla and H. Smith proved the following

Theorem (C. Delizia, A. H. Rhemtulla and H. Smith, 2000)

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The proof uses deep results by Lubotzky and Mann and the positive answer, due to Zel'manov, to the Restricted Burnside problem.

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Let G be a finitely generated locally graded E_k°-group. Then G is finite-by-(k-Engel) (in particular it is a finite extension of a k-Engel group).

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Given a group G, let $\Gamma_{X^\star}(G)$ be the simple graph whose vertices are all elements of G, and different vertices x and y are connected by an edge if $w(x,y) = 1$.

The group G is said to be an *$X^\star$ − group* if the graph $\Gamma_{X^\star}(G)$ has no infinite totally disconnected subgraphs.

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- By Lemma 2 the derived subgroup G' is finitely generated and, by induction, $\gamma_i(G)$ is finitely generated, for all $i > 0$.
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- By a result of Delizia and Nicotera (2001), if H is a finitely generated residually finite group in $E_2^\star$, then $H/Z_2(H)$ is finite, thus $\gamma_3(H)$ is finite.
- Since G/R is residually finite, we obtain $\gamma_3(G)/R$ finite and R is finitely generated.
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  Otherwise there exists a normal subgroup S of R, $S < R$ and of finite index in R.
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More generally, of course, we can formulate the following:

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If V is a variety defined by the law $w = 1$, is it true that $V^\star \subseteq V^\circ$?

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- By a result of Delizia and Nicotera (2001), if H is a finitely generated residually finite group in $E_2^\star$, then $H/Z_2(H)$ is finite, thus $\gamma_3(H)$ is finite.
- Since G/R is residually finite, we obtain $\gamma_3(G)/R$ finite and R is finitely generated.
- If $R = \{1\}$, we have done.
  Otherwise there exists a normal subgroup S of R, $S < R$ and of finite index in R.
  We can assume S normal in G. Then G/S is residually finite and $R \subseteq S$, a contradition.

More generally, of course, we can formulate the following:

Question

If V is a variety defined by the law $w = 1$, is it true that $V^\star \subseteq V^\circ$?

Let V be a variety of groups defined by the law $w(y_1, \ldots, y_n) = 1$.

Let $V^\sharp$ denote the class of all groups G in which, for any infinite subsets $X_1, \ldots, X_n$ of G, there exist $x_1 \in X_1, \ldots, x_n \in X_n$ such that $w(x_1, \ldots, x_n) = 1$.

Obviously $V \cup F \subseteq V^\sharp$, where F is the class of all finite groups.

It is known that for many varieties V and for many words w the equality $V \cup F = V^\sharp$ holds.

Theorem (P. L., M. Maj and A.H. Rhemtulla, 1995)

Let V is the variety of all nilpotent groups of class at most k defined by the law $[y_1, \ldots, y_{k+1}] = 1$. Then every $V^\sharp$-group is either finite or nilpotent of class at most k.

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For the variety of all k-Engel groups, with $k = 2, 3$ we have the following

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It is still open the following

Question (Problem 15.1 in The Kourovka notebook)

Does the equality $V \cup F = V^\sharp$ hold for any variety V and for any word w?

It is known that there exist classes of groups for which the previous equality holds.

Theorem (G. Endimioni, 1995)

Let V be a variety of groups defined by the law $w = 1$. Then an infinite $V^\sharp$-group G is a V-group in the following cases:

- *G is locally nilpotent;*
- *G is finitely generated and soluble, and every finitely generated soluble V-group is polycyclic;*
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