## What is Engel's law?

Engel's law is an observation in economics stating that, with a given set of tastes, as income rises, the proportion of income spent on food falls.

Wikipedia

# On Engel and positive laws
## What do they have in common?

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Bath, 13.08.2009

# Notation

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By Zelmanov's positive solution of the Restricted Burnside Problem all groups in $\mathfrak{B}_e$ are locally finite of exponent dividing $e$.

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# The Engel laws

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All known $n$-Engel groups are locally nilpotent.

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# Positive laws

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Note that *p.l.*- groups do not contain free subsemigroups.

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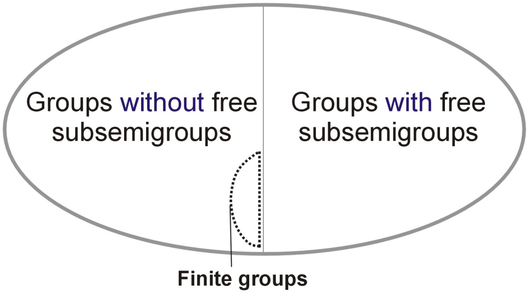

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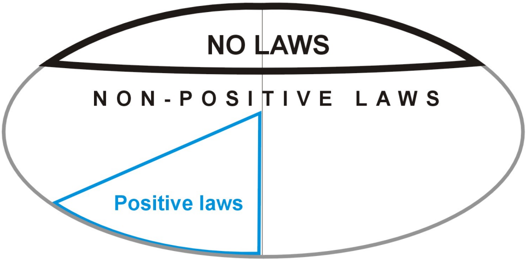

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**Corollary**

*Nilpotent-by-(finite exponent) groups satisfy positive laws.*

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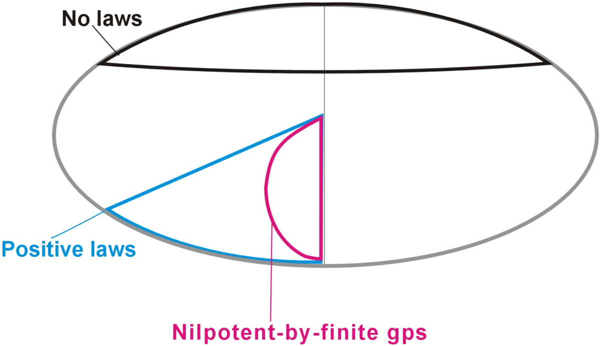

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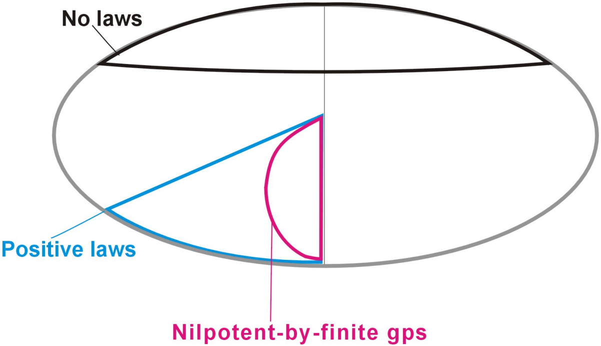


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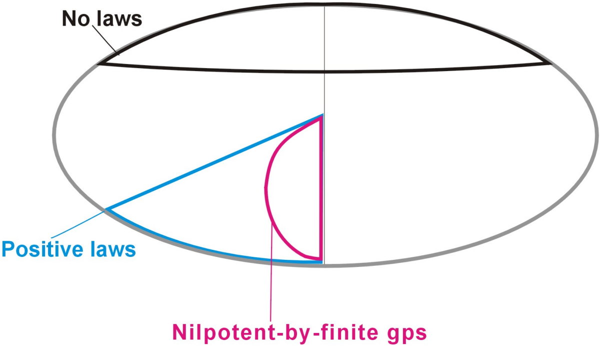


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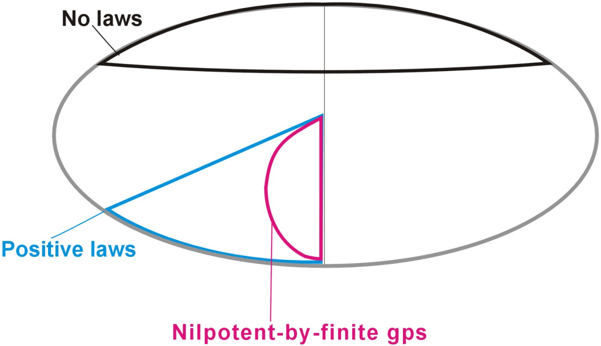


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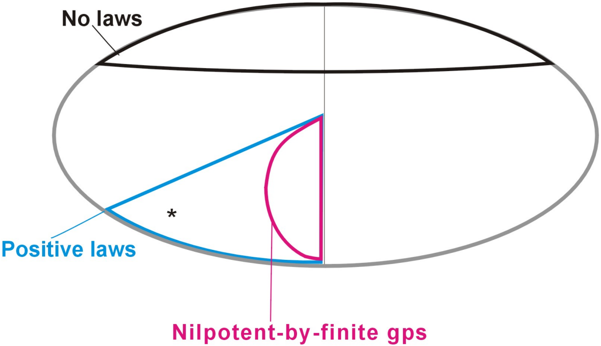


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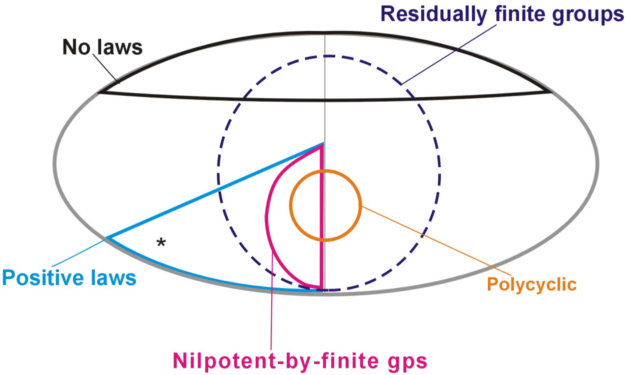

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It follows that a f.g. residually finite group satisfying an Engel law or a positive law is nilpotent-by-finite.

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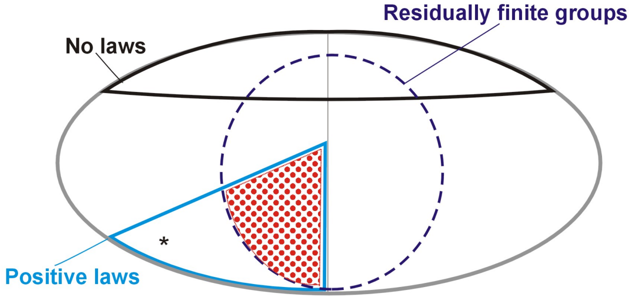

# Locally graded groups

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A group which has no f.g. infinite simple sections is locally graded.

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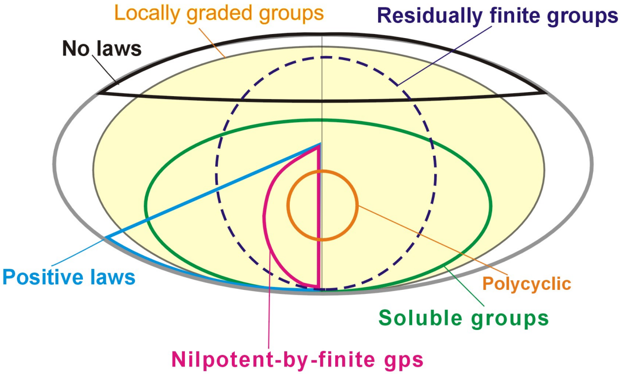

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It follows that a locally graded group

satisfying an Engel or a positive law is locally (soluble-by-finite).

. . . . . .

## Locally graded groups with Engel or positive laws are locally (soluble-by-finite) (*)

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**Corollary**

*If $G$ is a locally graded group satisfying an n-Engel law or a positive law of degree n then $G \in \mathfrak{N}_c\mathfrak{B}_e$, where $c, e$ depend on $n$ only.*

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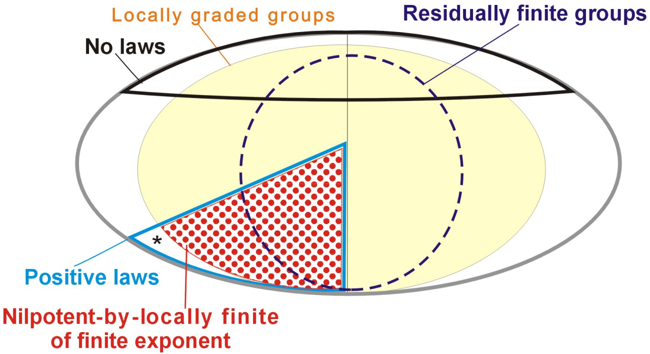

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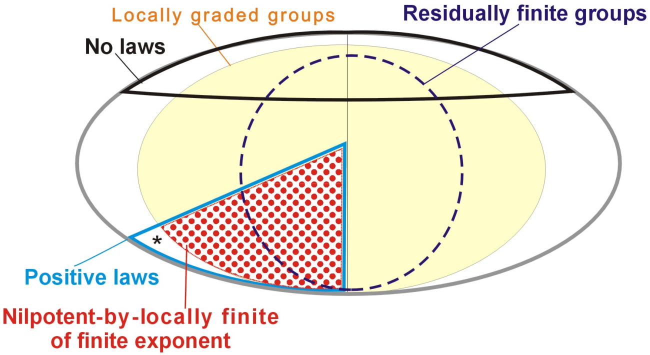


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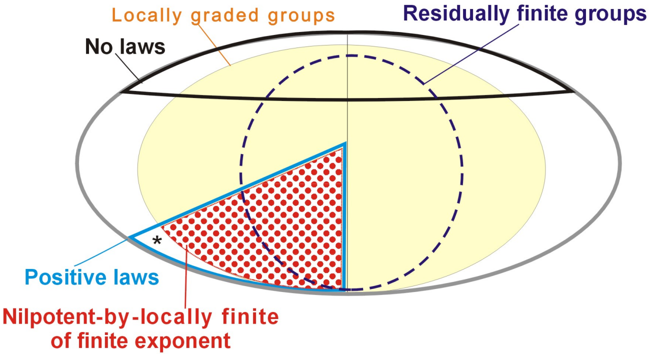


Question by R.G.Burns

*What do the Engel laws and positive laws have in common that forces finitely generated locally graded groups satisfying them to be nilpotent-by-finite?*

The answer is:

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The Engel laws and positive laws have the same so called Engel construction.

# Engel Construction of laws

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- The laws with construction $[x, y] \,\widetilde{\in}\, F''$ define varieties of groups with perfect commutator subgroups (i.e. $G' = G''$).

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We can show that every law has the General Engel Construction:

$$x^{k_0}[x,y]^{k_1}\,[x,\,{}_2y]^{k_2}\ldots\,[x,\,{}_ny]^{k_n}\;\widetilde{\in}\;E'$$

## $E_n = \langle\, [x,\ {}_i y],\ 0 \le i \le n \rangle, \quad E = \langle\, [x,\ {}_i y],\ 0 \le i \rangle.$

To show that every binary law has the General Engel Construction, we need a technical Lemma, which states:

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So we have

$$E_n = \langle x^{y^i},\ 0 \le i \le n\rangle \quad \text{and} \quad E = \langle x^{y^i},\ 0 \le i \rangle.$$

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Now we add $x^{k_0}$ to get the required construction.

## $\mathfrak{R}$-laws

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If $n = 1$ we have only one type of laws $x^k[x,y] \equiv 1$ defining varieties $\mathfrak{A}_k$.

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It can be shown that a positive law is the $\mathfrak{R}$-law.

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We say that a law is restraining if it provides the above property.

We show that a law is restraining if and only if it is an $\mathfrak{R}$-law.

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Recall: An $\mathfrak{R}$-law is the law implying a law with the Engel Construction $[x, y]^{k_1} [x, {}_2y]^{k_2} \dots [x, {}_{n-1}y]^{k_{n-1}} [x, {}_ny] \mathrel{\widetilde{\in}} E'_{n-1}$,

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**Theorem**

*A law $w\equiv 1$ is an $\mathfrak{R}$-law if and only if in every group $G$ satisfying this law*
*for all $g,h\in G$ the subgroup $\langle g^{h^i}, i\in\mathbb{N}\rangle$ is finitely generated.*

## $[x, {}_n y] \,\widetilde{\in}\, E_{n-1}$.

**Proof.**

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**Proof.** If use $[x, y^n] \in E_{n-1}[x, {}_n y]$ then we get $[x, y^n] \mathrel{\widetilde{\in}} E_{n-1}$, $x^{y^n} \mathrel{\widetilde{\in}} E_{n-1}$.

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if change $y \rightarrow y^{-1}$, then $x \,\widetilde{\in}\, \langle\, x^y,\ x^{y^2},\ \ldots, x^{y^{(n-1)}}, x^{y^n} \rangle$. Let $G$ be a relatively free group, freely generated by $a, b, \ldots$, satisfying an $\mathfrak{R}$-law, then $a \in \langle\, a^b,\ a^{b^2},\ \ldots, a^{b^{(n-1)}}, a^{b^n} \rangle$.

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Now use the fact, that $a, b$ are the free generators.

**What do the Engel laws and positive laws have in common?**

## Engel laws and positive laws are the $\mathfrak{N}$-laws

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1976, S. Rosset: if $G$ is a finitely generated group and
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**Proof**
There is a finite subnormal series from $G$ to $N$ with cyclic factors and by repeated application of result $(ii)$ we obtain that $N$ is finitely generated.

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**Theorem**

*A law is an $\mathfrak{R}$-law if and only if every f.g. group $G$ satisfying this law has its commutator subgroup $G'$ finitely generated.*

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**Lemma (cf. 2003, Burns and Medvedev )**

*If every* f.g. metabelian *group satisfying a law* $w \equiv 1$ *is nilpotent-by-finite then:*
*every* f.g. residually finite *group satisfying the law* $w \equiv 1$ *is nilpotent-by-finite.*
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*We show that*
*Every f.g. locally graded group satisfying an $\mathfrak{R}$-law is nilpotent-by-finite.*

# Every f.g. residually finite group satisfying an $\mathfrak{R}$-law is nilpotent-by-finite

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Then $R \subseteq K$, which contradicts to $K \subseteq T \subsetneq R$.

## Engel laws and positive laws are the $\mathfrak{K}$-laws

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## Corollary

*For every $\mathfrak{R}$-law there exist positive integers $c$ and $e$ depending only on the law, such that every locally graded group satisfying this law lies in the product variety $\mathfrak{N}_c\mathfrak{B}_e$.*

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**Problem** Is there an $\mathfrak{R}$-law that implies neither positive nor Engel law?

# Special kind of $\mathfrak{R}$-laws

# The construction $[x,\ _n y] \,\widetilde{\in}\, E_{n-1}$ defines the $\mathfrak{R}$-laws.

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We consider the laws called $L_n$ of the form

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$(ii)$ If there exist a non-abelian finite group satisfying $L_n$. Take such a group $G$ of the smallest order. By Miller and Moreno result (1903), a finite group $G$, all whose proper subgroups are abelian, is metabelian. Hence $G$ must be abelian, a contradiction.

So every law $L_n$ of the form $[x, y] \equiv [x, {}_n y]$, $n > 1$ is either abelian or pseudo-abelian.

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For $n = 2$ and $n = 3$ we used the commutator identities (the equality in $F$) saying that $[x, {}_n y^{-1}]$ is conjugate to $[x, {}_n y]^{(-1)^n}$.

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We conjecture that for $n>3$ the law $[x,y]\equiv[x,{}_n y]$ need not be abelian.

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We conjecture that for $n>3$ the law $[x, y] \equiv [x,\, _ny]$ need not be abelian.

**THANK YOU**

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