Hamiltonian cycles in the generating graphs of finite groups

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What is the talk about?

This talk is based on the following draft.

Authors:

- Thomas Breuer,
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- Gábor P. Nagy.

Title: Hamiltonian cycles in the generating graphs of finite groups.

The generating graph

Definition (Notation)

Let G be a non-trivial finite group.
Let $\Gamma(G)$ be the (simple) graph defined on the non-trivial elements of G in such a way that two distinct vertices are connected by an edge if and only if they generate G.

Why is the generating graph interesting?

1. Solvable groups

The clique number of a finite simple graph is the size of a largest complete subgraph of the graph. The chromatic number of a finite simple graph is the smallest number of colors needed to color the vertices of the graph in such a way that the endpoints of every edge receive different colors.

Theorem (Lucchini, M, 2008).

Let G be a finite group of Fitting height at most 2. Then the clique number and the chromatic number of $\Gamma(G)$ are equal.

Conjecture (Lucchini, M, 2008).

Let G be a finite solvable group. Then the clique number and the chromatic number of $\Gamma(G)$ are equal.

Why is the generating graph interesting?

2. Finite simple groups

Theorem (Steinberg, 1962; Aschbacher, Guralnick, 1984).

Every finite simple group can be generated by 2 elements.

What can be said about $\Gamma(G)$ when G is a finite simple group?

In the literature, many theorems about finite simple groups can be phrased in terms of the generating graph.

Theorem (Guralnick, Kantor, 2000).

If G is a finite simple group of order larger than 2, then there is no isolated vertex in $\Gamma(G)$.

Theorem (Guralnick, Shalev, 2003).

For every sufficiently large finite simple group G the diameter of the graph $\Gamma(G)$ is at most 2.

Theorem (Breuer, Guralnick, Kantor, 2008).

If G is a finite simple group of order larger than 2, then the diameter of the graph $\Gamma(G)$ is at most 2.

The conjecture of Breuer, Guralnick, and Kantor

The previous three theorems point in one direction...

Conjecture (Breuer, Guralnick, Kantor, 2008).

Let G be a finite group of order larger than 2. Then the following three conditions are equivalent.

1. G/N is cyclic for every non-trivial normal subgroup N of G;
2. there is no isolated vertex in $\Gamma(G)$;
3. the diameter of $\Gamma(G)$ is at most 2.

Burness, Guest, and Guralnick are working on this conjecture. They have reduced the problem to the case where G is almost simple.

For which finite groups G does there exist a Hamiltonian cycle in $\Gamma(G)$?

(A Hamiltonian cycle in a graph is a cycle that visits each vertex exactly once.)

Solvable groups

Theorem (Breuer, Guralnick, Lucchini, M, Nagy, 2009).

Let G be a finite solvable group of order larger than 3. Then the following four conditions are equivalent.

1. G/N is cyclic for every non-trivial normal subgroup N of G;
2. there is no isolated vertex in $\Gamma(G)$;
3. the diameter of $\Gamma(G)$ is at most 2;
4. there is a Hamiltonian cycle in $\Gamma(G)$.

Conjecture.

We may remove the word solvable from the statement of the above theorem. (The implications 4. implies 2. implies 1. are trivial.)

Pósa's criterion

For solvable groups G it is somewhat easy to exhibit a Hamiltonian cycle in $\Gamma(G)$ (provided that a Hamiltonian cycle exists). But this is not always the case for finite groups in general.

Theorem (Pósa).

Let Γ be a simple graph on m vertices whose sequence of vertex degrees is $d_1 \leq \ldots \leq d_m$. If $d_k \geq k+1$ for every positive integer k less than $m/2$, then there exists a Hamiltonian cycle in Γ.

Finite simple groups

Theorem (Breuer, Guralnick, Lucchini, M, Nagy, 2009).

For every sufficiently large finite simple group G the graph $\Gamma(G)$ satisfies Pósa's criterion hence there exists a Hamiltonian cycle in $\Gamma(G)$.

The proof relies on the following papers:

- Babai-Hayes (2006);
- Fulman-Guralnick (2008);
- Guralnick-Kantor (2000);
- Liebeck-Shalev (1996).

Chvátal's criterion

If G is a finite group having a subgroup of index 2, then there is little chance that the graph $\Gamma(G)$ satisfies Pósa's criterion. In this case we use (a weaker form of) Chvátal's criterion.

Theorem (weaker form) (Chvátal).

Let Γ be a graph on m points with sequence of vertex degrees $d_1 \leq \ldots \leq d_m$. Suppose that we have $m - k \leq d_{m-k}$ for all positive integers k with $k < m/2$. Then there exists a Hamiltonian cycle in Γ.

The Bondy-Chvátal closure

Pósa's and Chvátal's criteria are often combined with the following combinatorial tool.

Definition

Let Γ be a simple graph on m points. Let the degree of a vertex v be denoted by $d(v)$. The Bondy-Chvátal closure of the graph Γ is the graph $cl(\Gamma)$ obtained from Γ in such a way that there is an edge between vertices u and v if and only if there is an edge between u and v in Γ or there is no edge between u and v in Γ but $d(u) + d(v) \geq m$.

Theorem (Bondy, Chvátal, 1976).

The graph Γ contains a Hamiltonian cycle if and only if the Bondy-Chvátal closure $cl(\Gamma)$ contains a Hamiltonian cycle.

Symmetric groups

Theorem (Breuer, Guralnick, Lucchini, M, Nagy, 2009).

For every sufficiently large symmetric group S_n the graph $cl(cl(cl(\Gamma(S_n))))$ satisfies Chvátal's criterion hence there exists a Hamiltonian cycle in $\Gamma(S_n)$.

The proof relies on the following papers:

- Babai-Hayes (2006);
- Breuer-Guralnick-Kantor (2008).

Wreath products

Theorem (Breuer, Guralnick, Lucchini, M, Nagy, 2009).

Let n be a prime power. For every sufficiently large nonabelian finite simple group S (independent of n) the graph $cl(cl(\Gamma(S \wr C_n)))$ satisfies Pósa's criterion hence there exists a Hamiltonian cycle in $\Gamma(S \wr C_n)$.

The proof relies on the following papers:

- Babai-Hayes (2006);
- Fulman-Guralnick (2008);
- Guralnick-Kantor (2000).

Sporadic simple groups

The character-theoretic computational methods of the Breuer-Guralnick-Kantor (2008) paper helped obtain the following result.

Theorem (Breuer, Guralnick, Lucchini, M, Nagy, 2009).

Let G be an almost simple group whose socle is a sporadic simple group which is different from the Monster. Then the graph $cl(cl(\Gamma(G)))$ satisfies Pósa's criterion and hence there exists a Hamiltonian cycle in $\Gamma(G)$.

We intend to deal with the case when G is the Monster group.

Small groups

The following is a weaker form of a conjecture we have already mentioned.

Conjecture (Breuer, Guralnick, Lucchini, M, Nagy, 2009).

Let G be a finite group of order larger than 3. Then $\Gamma(G)$ contains a Hamiltonian cycle if and only if G/N is cyclic for every non-trivial normal subgroup N of G.

Theorem (Breuer, Guralnick, Lucchini, M, Nagy, 2009).

This conjecture is true for

- all groups of orders at most 10^5;
- all almost simple groups of orders at most 10^6;
- all nonabelian finite simple groups of orders at most 10^7.

Thank you for your attention!