# Extending the Kegel-Wielandt theorem through $\pi$-decomposable groups

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We are interested in viewing nilpotent groups as a particular case of groups in a lattice-formation and so as $\pi$-decomposable groups, for a set of primes $\pi$.

# $\pi$-decomposable groups

## Notation

For any group $X$ and any set of primes $\pi$, denote

- $O_\pi(X)$ the largest normal $\pi$-subgroup in $X$,
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## Definition

A group $X$ is said to be $\pi$-decomposable if

$$X = O_\pi(X) \times O_{\pi'}(X) = X_\pi \times X_{\pi'}$$

is the direct product of a $\pi$-subgroup and a $\pi'$-subgroup, where $\pi' = \mathbb{P} - \pi$.

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## Example

Let $G = A_5$, the alternating group of degree 5. Then:

$$G = AB, \quad A = G_{\{2,3\}} \in \mathrm{Hall}_{\{2,3\}}(G), \quad B = G_5 \in \mathrm{Syl}_5(G)$$

Let $\pi = \{2, 3\}$. Then $G = AB$ is the product of a $\pi$-group and a $\pi'$-group, but $G$ is not $\pi$-separable.

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## Question

- When is it true that

$$A_\pi B_\pi = B_\pi A_\pi$$

and so $G$ posseses Hall $\pi$-subgroups?

# First approach: one of the factors is a $\pi$-group

**Lemma**

*Let the group $G = AB$ be the product of a $\pi$-decomposable subgroup $A$ and a $\pi$-subgroup $B$. Then the following statements are equivalent:*

*(i)* $A_\pi = O_\pi(A) \leq O_\pi(G)$;

*(ii)* $A_\pi B = BA_\pi$ *is a Hall* $\pi$*-subgroup of* $G$.

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**Theorem**

*Let $\pi$ be a set of odd primes. Let the group $G = AB$ be the product of a $\pi$-decomposable subgroup $A$ and a $\pi$-subgroup $B$. Then $A_\pi = O_\pi(A) \leq O_\pi(G)$.*

- L. S. Kazarin, A. Martínez-Pastor and M.D. Pérez-Ramos. On the product of a $\pi$-group and a $\pi$-decomposable group, J. Algebra, (2007).

## Corollary

*Let $\pi$ be a set of odd primes. Let the group $G = AB$ be the product of a $\pi$-decomposable subgroup $A$ and a $\pi$-subgroup $B$. Then the composition factors of $G$ belong to one of the following types:*

1) *$\pi$-groups,*
2) *$\pi'$-groups,*
3) *groups in the following set:*

$$\{\, M_{11},\ M_{23},\ L_2(q),\ \text{with either } q = 29 \text{ or } 3 < q \not\equiv 1 \pmod 4;$$
$$L_r(q),\ q \text{ an odd prime},\ (r, q-1) = 1;\ A_r,\ r \geq 5,\ \text{a prime}\ \}$$

*In particular, if none of these groups is involved in $G$, then the group is $\pi$-separable.*

## Example

$$G = AB,\ A = A_\pi \times A_{\pi'},\ B = B_\pi,\ 2 \in \pi \ \not\Rightarrow\ O_\pi(A) \not\leq O_\pi(G)$$

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Let

$$G \cong L_2(q),\ q = 2^n,$$

with $q + 1 = 2^n + 1$ divisible by two distinct primes (this happens if $n \neq 3$ and $2^n + 1$ is not a Fermat prime). Then

$$G = AB,\ A \cong C_{q+1},\ B = N_G(G_2),\ G_2 \in \mathrm{Syl}_2(G).$$

Let $r$ be a prime dividing $q + 1$ and take $\pi = \pi(N_G(G_2)) \cup \{r\}$. Then $A = A_\pi \times A_{\pi'}$, $B$ is a $\pi$-group, but $O_\pi(A) \not\leq O_\pi(G)$.

# Coprime orders

**Theorem**

*Let $\pi$ be a set of odd primes. Let the group $G = AB$ be the product of two $\pi$-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Assume in addition that $(|A_{\pi'}|, |B_{\pi'}|) = 1$. Then $A_\pi B_\pi = B_\pi A_\pi$.*

In particular, the result holds in the significant case when $(|A|, |B|) = 1$.

# Known results

**Theorem**

*Let $G = AB$ be a group satisfying:*

(1) $A = A_2 \times A_{2'}$ *is 2-decomposable.*

(2) *$B$ is nilpotent and $|B|$ is odd.*

(3) $(|A|, |B|) = 1$

*Then $G$ is soluble.*

- Ya. G. Berkovich, Generalization of the theorems of Carter and Wielandt, Sov. Math. Dokl., (1966).

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- P. J. Rowley, The $\pi$-separability of certain factorizable groups, Math. Z. (1977).

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Previously Kazarin had obtained that $A_{2'} = O_{2'}(A) \leq O_{2'}(G)$ under the above hypotheses.

- L. S. Kazarin, Criteria for the nonsimplicity of factorable groups, Izv. Akad. Nauk SSSR, (1980).

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*Then* $A_{2'} = O_{2'}(A) \leq O_{2'}(G)$.

- L. S. Kazarin, The product of a 2-decomposable group and a group of odd order, Problems in group theory and homological algebra (1983).

## Theorem

*Let $\pi$ be a set of odd primes. Let the group $G = AB$ be the product of a $\pi$-decomposable subgroup $A$ and a $\pi$-subgroup $B$. Then $A_\pi = O_\pi(A) \leq O_\pi(G)$.*

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# The soluble case with $\pi$ a set of odd primes

## Theorem

*Let $\pi$ be a set of odd primes. Let the group $G = AB$ be the product of two $\pi$-decomposable soluble subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Then $A_\pi B_\pi = B_\pi A_\pi$ and this is a Hall $\pi$-subgroup of $G$.*

- L. S. Kazarin, A. Martínez-Pastor and M.D. Pérez-Ramos. On the product of two $\pi$-decomposable soluble groups, Publ. Mat., (2009).

# The soluble case with $2 \in \pi$

**Theorem**

*Let $\pi$ be a set of primes with $2 \in \pi$. Let the group $G = AB$ be the product of two soluble $\pi$-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Assume that the following simple groups are not involved in G:*

*(i) $L_2(2^n)$, $n \geq 2$, except if either $n = 3$ or $q = 2^n + 1 > 5$ is a Fermat prime,*

*(ii) $L_2(q)$, $q > 3$ odd, except if q is a Mersenne prime.*

*Then $A_\pi B_\pi = B_\pi A_\pi$ and this is a Hall $\pi$-subgroup of G.*

- L.S. Kazarin, A. Martínez-Pastor, M.D. Pérez-Ramos. On the product of two $\pi$-decomposable soluble groups, Publ. Mat., (2009).

## Example

$$\left.\begin{array}{c} G = AB,\ A = A_\pi \times A_{\pi'},\ B = B_\pi \times B_{\pi'},\ 2 \in \pi \\ A, B \text{ soluble} \end{array}\right\} \not\Rightarrow A_\pi B_\pi = B_\pi A_\pi$$

## Example

$$\left.\begin{array}{c} G = AB,\ A = A_\pi \times A_{\pi'},\ B = B_\pi \times B_{\pi'},\ 2 \in \pi \\ A, B \text{ soluble} \end{array}\right\} \not\Rightarrow A_\pi B_\pi = B_\pi A_\pi$$

Let

$$G \cong PGL_2(q),\ q = p^n > 3 \text{ odd },\ q \text{ not a Mersenne prime.}$$

Then

$$G = AB,\ A \cong C_{q+1},\ B = N_G(G_p),\ G_p \in \mathrm{Syl}_p(G).$$

Take $\pi = \pi(N_G(G_p))$ and notice that $2 \in \pi$. Then $A = A_\pi \times A_{\pi'}$ is a $\pi$-decomposable group, $B$ is a $\pi$-group, but $A_\pi B$ is not a subgroup (since $q + 1$ is not a power of 2).

## Corollary

*Let $\pi$ be a set of primes. Let $G = AB$ be the product of two soluble $\pi$-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Assume that the following simple groups are not involved in $G$:*

*(i) $L_2(2^n)$, $n \geq 2$, except if either $n = 3$ or $q = 2^n + 1 > 5$ is a Fermat prime,*

*(ii) $L_2(q)$, $q$ odd, except if $q$ is a Mersenne prime.*

*Then the composition factors of $G$ are of the following types:*

*1) $\pi$-groups,*

*2) $\pi'$-groups,*

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$$\{\, L_2(2^n), n \geq 2, \text{ with either } n = 3 \text{ or } q = 2^n + 1 > 5 \text{ a Fermat prime}, \\ L_2(q), q > 3, q \text{ a Mersenne prime}, \ L_3(3), \ M_{11} \}$$

# The soluble case

## Corollary

*Let $\pi$ be a set of primes. Let the group $G = AB$ be the product of the two soluble $\pi$-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ and assume that the simple groups*

$$L_2(q),\ q > 3,\ L_3(3) \text{ and } M_{11}$$

*are not involved in $G$.*
*Then the group $G$ is $\pi$-separable.*

## Preliminary basic results

### Lemma

*Let the group $G = AB$ be the product of the subgroups $A$ and $B$ and let $A_0 \trianglelefteq A$ and $B_0 \trianglelefteq B$. If $A_0B_0 = B_0A_0$, then*

$$A_0^g B_0 = B_0 A_0^g \text{ for all } g \in G.$$

*If, in addition, $A_0$ and $B_0$ are $\pi$-groups for a set of primes $\pi$, then $[A_0, B_0]$ is a subnormal $\pi$-subgroup of $G$. In particular, if $O_\pi(G) = 1$, then $[A_0^G, B_0^G] = 1$.*

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- This result is applicable if $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ are $\pi$-decomposable, for $A_0 = A_\pi$ and $B_0 = B_\pi$.

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- This result is applicable if $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ are $\pi$-decomposable, for $A_0 = A_\pi$ and $B_0 = B_\pi$.

**Lemma**

*Let the group $G = AB$ be the product of the subgroups $A$ and $B$. Then for each prime $p$ there exist Sylow $p$-subgroups $A_p$ of $A$ and $B_p$ of $B$ such that $A_p B_p$ is a Sylow $p$-subgroup of $G$.*

# Our conjecture

## Conjecture

Let the group $G = AB$ be the product of two $\pi$-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$.
Let $\pi$ be a set of odd primes. Then

$$A_\pi B_\pi = B_\pi A_\pi$$

and this is a Hall $\pi$-subgroup of $G$.

# The minimal counterexample

## Proposition

*Let $\pi$ be a set of odd primes. Assume that the group $G = AB$ is the product of two $\pi$-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, and $G$ is a counterexample of minimal order to the assertion $A_\pi B_\pi = B_\pi A_\pi$.*
*Then $G$ has a unique minimal normal subgroup $N$, which is a non-abelian simple group.*
*Moreover $A_\pi \neq 1$, $B_\pi \neq 1$,*

$$G = AN = BN = AB, \ (|A_{\pi'}|, |B_{\pi'}|) \neq 1, \ A_{\pi'} \cap B_{\pi'} = 1,$$

*and $\pi(G) = \pi(N)$.*

# The positive answer

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Let the group $G = AB$ be the product of two $\pi$-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$.
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We have recently proved that:

The conjecture is true!

- L. S. Kazarin, A. Martínez-Pastor and M.D. Pérez-Ramos. On the product of two $\pi$-decomposable groups, preprint.