Schur multipliers of 4-Engel groups

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The Schur multiplier

Let G be a finite group. The second cohomology group $H^2(G, \mathbb{C}^\times)$ is said to be the **Schur multiplier** of G.

This group has its origins in the work of Schur (1904) on projective representations of groups. Applications include central extensions of groups.

If G is finite, then $H^2(G, \mathbb{C}^\times) \cong H_2(G, \mathbb{Z})$. If G is infinite, then we say that $H_2(G, \mathbb{Z})$ is the Schur multiplier of G.

Notation: $M(G)$.

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5. Bounds for the rank of $M(G)$.
6. Bounds for the exponent of $M(G)$.

The exponent of $M(G)$ – an application

A subfield K of $\mathbb{C}$ is a **weak projective splitting field** for G if every irreducible projective representation $\rho : G \to \mathrm{GL}(m, \mathbb{C})$ of G over $\mathbb{C}$ there exists a map $\lambda : G \to \mathbb{C}^\times$ and a matrix $A \in \mathrm{GL}(m, \mathbb{C})$ such that

$$\lambda(g)A\rho(g)A^{-1} \in \mathrm{GL}(m, K) \quad \text{for all} \quad g \in G.$$

(ρ is **projectively realizable** over K).

Theorem (Opolka 1981)

Let G be a finite group. Then

$$\mathbb{Q}(\zeta_e), \quad \textit{where} \quad e = \exp[G, G] \cdot \exp M(G),$$

is a weak projective splitting field for G.

The exponent of $M(G)$ – Schur's bound

If G is a finite group, then $M(G)$ is a finite group of exponent dividing $|G|$.

Theorem (Schur 1904)

Let G be a finite group and $e = \exp M(G)$. Then e^2 divides $|G|$.

This bound is best possible, since

$$M(\mathbb{Z}/e\mathbb{Z} \oplus \mathbb{Z}/e\mathbb{Z}) \cong \mathbb{Z}/e\mathbb{Z}.$$

1. Does $\exp M(G)$ divide $\exp G$ for every finite group G?
2. Can we bound $\exp M(G)$ in terms of $\exp G$ for every finite group G?
3. Determine precise (or reasonable) bounds for $\exp M(G)$.

Does $\exp M(G)$ divide $\exp G$?

It was a longstanding open problem as to whether $\exp M(G)$ divides $\exp G$ for every finite group G.

Let G be a group of finite exponent. The answer is affirmative if G belongs to one of the following classes:

- Abelian groups.
- Finite simple groups; CFSG.
- Groups of exponent 2.
- Groups of exponent 3; M. R. Jones 1973.
- Groups of exponent 6.

Does $\exp M(G)$ divide $\exp G$?

More good classes of groups:

- Groups that are nilpotent of class ≤ 3; M. R. Jones 1973.
- Powerful p-groups; A. Lubotzky and A. Mann 1987.
- p-groups with potent filtrations; M 2009.
- p-groups of maximal class; M 2009.

The answer is false in general; there exists a group G of order 2^{68}, exponent 4 and class 4, with $\exp M(G) = 8$.

Question

Does there exist a group of odd order such that $\exp M(G)$ does not divide $\exp G$?

Some bounds for $\exp M(G)$

Theorem (M 2007)

Let G be a group of exponent e.

- *If G is nilpotent of class ≤ 4, then* $\exp M(G)$ *divides* $2e$.
- *Let G be nilpotent of class $c \geq 2$. Then* $\exp M(G)$ *divides* $e^{\min\{2\lfloor \log_2 c \rfloor, \lceil c/2 \rceil\}}$.
- *Let G be metabelian. Then* $\exp M(G)$ *divides* e^2.
- *If $e = 4$, then* $\exp M(G)$ *divides* 8.

Theorem (Lubotzky and Mann 1987)

Let G be a finite p-group of rank r and exponent p^e. Then $\exp M(G)$ *divides* p^{e+rn}, *where*

$$n = \begin{cases} \lceil \log_2 r \rceil & : \quad \text{if } p > 2 \\ \lceil \log_2 r \rceil^2 + 1 & : \quad \text{if } p = 2 \end{cases}.$$

Can $\exp M(G)$ be bounded in terms of $\exp G$?

By Schur's result and Zelmanov's solution of the RBP there exists a function $f(r,e)$ such that $\exp M(G)$ divides $f(r,e)$ for every finite r-generator group G of exponent e.

Theorem (M 2007)

There exists a function $\mathfrak{m}(e)$ such that $\exp M(G)$ divides $\mathfrak{m}(e)$ for every locally finite group of exponent e.

The proof yields a rather nasty bound:

$$\mathfrak{m}(e) \leq |R(2,e)| \leq \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{e^{e^{e}}}.$$

n-Engel groups

A group G is n-**Engel** if it satisfies the law $[x, \underbrace{y, \ldots, y}_{n}] = 1$.

Proposition

For every positive integer n there exists an integer $m = m(n)$ such that if p is a prime larger than m, and G is a finite n-Engel p-group, then $\exp M(G)$ divides $\exp G$.

Theorem (M 2008)

Let G be a 4-Engel group of exponent e. Then $\exp M(G)$ divides $10e$. If e is not divisible by 2 or 5, then $\exp M(G)$ divides e.

In general, the factor 2 cannot be eliminated. For 5 this is not clear.

Theorem (M 2008)

Let G be a 3-Engel group of exponent e. Then $\exp M(G)$ divides e.

Reduction and strategy

J. Wiegold: The Schur multiplier: an elementary approach, in *Proceedings of Groups St Andrews 1981.*

- We may assume that G is finitely generated, hence finite.
- We may assume that G is a finite p-group.

A group H is said to be a **covering group** of a group G if there exists a subgroup Z of H such that

(i) $Z \leq Z(H) \cap [H, H]$,

(ii) $Z \cong M(G)$,

(iii) $G \cong H/Z$.

Thus it suffices to consider $[H, H]$. There are some limitations.

Centre-by-4-Engel groups

Theorem

Let H be a finite p-group and let Z be a central subgroup of H such that $G = H/Z$ is a 4-Engel group. If $p \neq 2, 5$, then

$$[H, H]^{p^e} \leq [H^{p^e}, H].$$

For $p = 2$ or $p = 5$ we have that $[H, H]^{p^{e+1}} \leq [H^{p^e}, H]$.

Theorem

Let H be a finite p-group and let Z be a central subgroup of H such that $G = H/Z$ is a 3-Engel group. Then $[H, H]^{p^e} \leq [H^{p^e}, H]$.

Basic idea: Suppose $[H^{p^e}, H] = 1$,prove $[x, y]^{p^e} = 1$, and $(\omega[x, y])^{p^{e+\epsilon}} = \omega^{p^{e+\epsilon}}$ for all $\omega \in [H, H]$.

Reduction to the case $p \in \{2,3,5\}$

Traustason (1995) proved that if G is a 4-Engel group without elements of order 2, 3 or 5, then G is nilpotent of class ≤ 7.

Consequently, H is regular for $p > 7$. In this case we are done.

If $p = 7$, H is also regular. This follows from the identity

$$\begin{aligned}(xy)^7 = {} & x^7 y^7 [y,x]^{21}[y,x,x]^{35}[y,x,y]^{91}[y,x,x,x]^{35}[y,x,y,x]^{175} \\ & \cdot [y,x,y,y]^{175}[y,x,x,x,x]^{21}[y,x,x,x,y]^{-42}[y,x,y,x,x]^{231} \\ & \cdot [y,x,y,x,y]^{-546}[y,x,y,y,x]^{917}[y,x,y,y,y]^{189} \\ & \cdot [y,x,y,x,x,x]^{-7}[y,x,y,y,x,x]^{3031}[y,x,y,y,x,y]^{28} \\ & \cdot [y,x,y,x,x,y,x]^{-21}[y,x,y,y,x,x,x]^{42} \\ & \cdot [y,x,y,y,x,x,y]^{-3269}\end{aligned}$$

that holds in H. The rest of the proof is commutator calculus in 2-, 3-, and 5-groups (not to be done in public).

Computational part of the proof

Let H be a centre-by-4-Engel p-group.

Using the Nilpotent quotient algorithm, one can construct:

- The free 3-generator 4-Engel group; Nickel 1999.
- The free 2-generator centre-by-4-Engel group.

Relations satisfied by these groups can be used to find (some) identities satified by H.

Construction of the free 3-generator centre-by-4-Engel group (nilpotent of class 10) is out of reach of current computational functionality.

Lemma (Nickel 1999)

Let E_r be a free 4-Engel group on r generators.

(a) *E_2 is torsion-free and nilpotent of class 6.*

(b) $\gamma_8(E_3)^{30} = \gamma_9(E_3)^3 = \gamma_{10}(E_3) = 1.$

The power commutator structure

Lemma (Havas and Vaughan-Lee 2007)

Let H be a centre-by-4-Engel 5-group.

(a) *Every 2-generator subgroup of H is nilpotent of class ≤ 7.*

(b) *Every 3-generator subgroup is nilpotent of class ≤ 9.*

(c) *If $r > 3$, every r-generator subgroup of H is nilpotent of class $\leq 2r + 1$.*

Lemma

Let $p \in \{2, 3\}$ and let G be a 4-Engel p-group. If $\exp G = p^e$, then $\mho_{e-1}(\gamma_4(\langle a, b, c\rangle)) = 1$ for all $a, b, c \in G$.

Lemma

Let G be a 3-Engel group of exponent 2^e and $a, b, c \in G$. Then $\mho_{e-1}(\gamma_3(\langle a, b, c\rangle)) = 1$.

$p = 5$: potential example of G with $\exp M(G) \nmid \exp(G)$

Let $G_{e,g}$ be the largest 4-Engel quotient of

$$\left\langle a_1, \dots, a_g, b_1, \dots, b_g \mid \text{ exponent } 5^e, \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle .$$

Let H be a covering group of $G_{e,g}$. We have an exact sequence

$$1 \longrightarrow M(G_{e,g}) \longrightarrow [H, H] \longrightarrow [G_{e,g}, G_{e,g}] \longrightarrow 1.$$

Let $\bar{a}_i$ and $\bar{b}_i$ be preimages of a_i and b_i in H. Then

$$\omega = \prod_{i=1}^{g} [\bar{a}_i, \bar{b}_i] \in M(G_{e,g}).$$

If $g > 2$, it might happen that $|\omega| > 5^e$.