

# Levi-properties in groups: The Engel connection

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Coverings of groups by subgroups characterize properties of groups and their structure.

For instance we have the following results:

Theorem (B. H. Neumann)

*A group  $G$  is covered by finitely many proper subgroups if and only if  $G$  has a finite noncyclic quotient.*

Theorem (R. Baer)

*A group  $G$  is covered by a finitely many abelian subgroups if and only if  $G$  is central-by-finite.*

# Introduction

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# Coverings by normal $\mathfrak{X}$ -subgroups

We can consider coverings of groups by systems of normal subgroups with a common group theoretic property:

- Finite coverings by proper normal subgroups with a common group theoretic property.
- Maximal subgroups with a common group theoretic property are normal.
- The normal closure of each element of the group has a common group theoretic property.
- Each of these coverings influence the group structure in decreasing amounts.
- Each of these covering are related to the Engel property.

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# Finite coverings by proper normal $\mathfrak{X}$ -subgroups

Results coupling the finite coverings to Engel groups.

Theorem (Brodie, Chamberlain, Kappe (1988))

*A group has a finite covering consisting of normal abelian subgroups if and only if it is central-by-finite 2-Engel group.*

Theorem (Brodie, M (2002))

*The following statements are equivalent for any group if  $n = 1, 2$  and for all  $n \geq 3$  if  $G$  is center-by-metabelian:*

- (a)  $G$  has a finite covering by subnormal abelian subgroups of defect at most  $n$ ;*
- (b)  $G$  is a central-by-finite  $(n + 1)$ -Engel group.*

We will extend the second theorem a bit later on.

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# Maximal $\mathfrak{X}$ -subgroups normal

Results coupling certain maximal subgroups normal to Engel groups.

## Theorem

*A group  $G$  has maximal abelian subgroups normal if and only if  $G$  is 2-Engel.*

## Theorem (Kappe, M (1990))

*Let  $G$  be a metabelian group. Under certain conditions for  $n \geq 2$  the following conditions are equivalent:*

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# Normal closure of each element is an $\mathfrak{X}$ -group

Let  $\mathfrak{X}$  be a class of groups. The derived class of groups  $L(\mathfrak{X})$  is the class of those groups in which the normal closure of each element in the group is an  $\mathfrak{X}$ -group.

The property of being in  $L(\mathfrak{X})$  is called the **Levi-property generated by  $\mathfrak{X}$** .

## Proposition

*The Fitting property is the Levi-property generated by  $\mathfrak{N}$ , the class of all nilpotent groups. Symbolically, if  $\mathfrak{F}$  is the class of Fitting groups, then  $\mathfrak{F} = L(\mathfrak{N})$ .*

## Theorem (Kappe, M (1990))

*A metabelian group  $G$  is bounded left Engel if and only if the normal closure of each element is nilpotent.*

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# Observations

By saying a group  $G$  is in  $L(\mathfrak{X})$  we need not gain much structural information about the group.

The worse case are the “radical” classes. For example it is not hard to show that for the property of being locally nilpotent we have

$$L\mathfrak{N} = L(L\mathfrak{N}).$$

Saying a property is a Levi-property is a weak statement.

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# Varieties of groups

Let  $\mathfrak{V}$  be a variety of groups.

Proposition (M. 1994)

*The class  $L(\mathfrak{V})$  is a variety of groups.*

Not all varieties are Levi-varieties.

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*Every proper subvariety of zero exponent of the 2-Engel groups is not a Levi-variety.*

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# Varieties of groups (cont.)

On the other hand, if the laws of  $\mathfrak{V}$  are nice (say recursively defined) then we can describe the laws of  $L(\mathfrak{V})$ .

## Theorem (M. 1994)

*If  $\mathfrak{V}$  can be described by outer commutator laws, then the laws of the variety  $L(\mathfrak{V})$  can be exactly described.*

So as a corollary we can “exactly” describe the Levi-property generated by  $\mathfrak{N}_c$ :

## Corollary

*The variety  $L(\mathfrak{N}_c)$  is exactly described by the law*

$$[x^{y_1}, x^{y_2}, \dots, x^{y_c}, x] = 1.$$

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# The $n$ -Engel property as a Levi-property

When can we characterize the  $n$ -Engel property as a Levi-property generated by  $\mathfrak{X}_n$ ?

How far can we go if  $\mathfrak{X}_n$  is a variety?

Each statement below is a consequence of the statement preceding it. For any group  $G$ :

- (i)  $a^G$  is nilpotent of class at most  $n$  for all  $a$  in  $G$  i.e.  $G \in L(\mathfrak{N}_n)$ ;
- (ii)  $a^G$  is  $n$ -Engel for all  $a$  in  $G$  i.e.  $G \in L(\mathfrak{E}_n)$ ;
- (iii)  $G$  is  $(n+1)$ -Engel.

We have the following inclusions

$$L(\mathfrak{N}_n) \leq L(\mathfrak{E}_n) \leq \mathfrak{E}_{n+1}.$$

All these inclusions are strict for  $n > 4$ .

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# The $n$ -Engel property as a Levi-property (cont)

N. Gupta, F. Levin (1980) construct a 4-Engel 5-group of class 7 with an element whose normal closure is nilpotent of class greater than 3.

Vaughan-Lee (2007) exhibits a 5-Engel 3-group of class 9 which has an element whose normal closure is exactly 5-Engel.

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Theorem (Levi (1942), W.P Kappe, L.-C. Kappe (1972))

*For  $n = 2$  and  $n = 3$  we have  $\mathfrak{E}_n = L(\mathfrak{N}_{n-1})$ .*

Theorem (Vaughan-Lee (2007))

*The  $\mathfrak{E}_4 = L(\mathfrak{E}_3)$ .*

Open Question

*For  $n > 4$  can the  $n$ -Engel property be characterized as a Levi-variety? Levi-property?*

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From the fact that 4-Engel is the Levi-property generated by 3-Engel we obtain the following:

## Theorem

*The following statements are equivalent for a group  $G$ :*

- (i)  $G$  is a central-by-finite 4-Engel group;*
- (ii)  $G$  is covered by a finite number of normal 3-Engel subgroups;*
- (iii)  $G$  is covered by a finite number of subnormal abelian subgroups of defect at most 3.*



# The $n$ -Engel property as a Levi-property

For some variations of solvability positive results hold.

- An metabelian  $n$ -Engel group  $G$  is in  $L(\mathfrak{N}_{n-1})$ . Kappe, M. (1990)
- A center-by-metabelian  $n$ -Engel group  $G$  is in  $L(\mathfrak{N}_{n-1})$ . Brodie, M. (2002)

Let  $G$  be in the “nearly center-by-metabelian” variety  $[[x_1, x_2, x_3], [x_4, x_5], x_6] = 1$ . N. Gupta, M. Newman (1989)

- A “nearly center-by-metabelian”  $n$ -Engel group  $G$  is in  $L(\mathfrak{N}_{n-1})$ . M (1999)

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# Areas for further investigation

- A better understanding of the groups in  $L(\mathfrak{E}_n)$ .

The solvable examples of groups not in  $L(\mathfrak{N}_{n-1})$  are all in  $L(\mathfrak{E}_n)$ . These groups are not even Fitting groups.

- Suppose for some  $n$  all  $n$ -Engel groups are in some Levi-property. Does this impose some structural restriction on these  $n$ -Engel groups?

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Restricting the nilpotency class of an  $n$ -Engel group forces it to be in a Levi-property:

Theorem (M 1999)

*Let  $n \geq 3$  and let  $G$  be a nilpotent group of class  $n + 2$ . Then  $G$  is  $n$ -Engel if and only if  $G$  is in  $L(\mathfrak{N}_{n-1})$ .*

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