An application of Magma to groups acting on trees

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A Kac-Moody group

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This is a rank 2, affine Kac-Moody group

Bass-Serre theory

Standard Borel and parabolics:

$$B = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{F}_q[[t^{-1}]]) \,\middle|\, c \equiv 0 \bmod (t^{-1}) \right\}$$

$$P_1 = \mathrm{SL}_2(\mathbb{F}_q[[t^{-1}]])$$

$$P_2 = \left\{ \begin{pmatrix} a & tb \\ c/t & d \end{pmatrix} \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{F}_q[[t^{-1}]]) \right\}$$

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X a graph:

- Vertices are all conjugates of P_1 and P_2
- Edge P—Q when $P \cap Q$ contains a conjugate of B

The tree X for $q = 2$

See Ulrich Görtz's website for a nice picture of this tree
`http://www.math.uni-bonn.de/people/ugoertz/tree1.ps`

Congruence graphs

Let $g \in \mathbb{F}_q[t]$ with degree n

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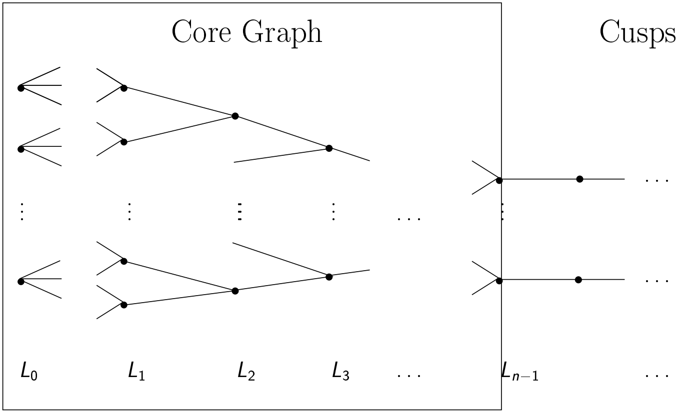
X_g is the coset graph of

$$H = \Gamma/\Gamma(g) \cong \mathrm{SL}_2(R) \qquad \text{for } R = \mathbb{F}_q[t]/(g)$$

wrt subgroups

$$H_0 = \mathrm{SL}_2(\mathbb{F}_q)$$
$$H_i = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a \in \mathbb{F}_q^\times, b \in R, \deg(b) \le i \right\}$$

Schematic of X_g



Magma computations

We computed these graphs in Magma using:

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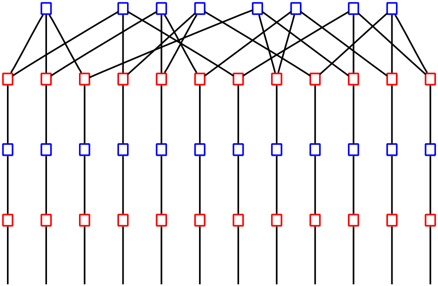
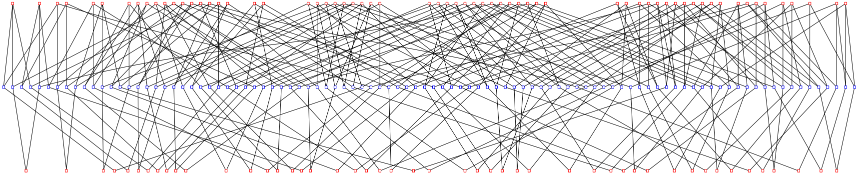
We found all X_g with less than $2^{16} - 1$ vertices
and several larger ones

X_g for $q = 2$ and $g(t) = t^2$



X_g for $q = 2$ and $g(t) = t^3$



Connectedness of X_g

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X_g is connected for all q and all $g \in \mathbb{F}_q[t]$

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Proof:
For $a \in R^\times$:

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -a^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \langle H_0, H_{n-1} \rangle$$

So all elementary matrices are in $\langle H_0, H_{n-1} \rangle$
So $H = \langle H_0, H_{n-1} \rangle$

The graph for PGL_2 (Morgenstern)

Let $H = \mathrm{PGL}_2(R)$
Take analogous subgroups $H_0, H_1, \ldots$
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Let $g(t) = t^n$
Let $\widetilde{D}_g$ be the subgraph of cosets of H_0 and H_1
Morgenstern claims this is a family of expanders (for each q)

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$\widetilde{X}_g$ is the wrong graph

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$$\Gamma/\Gamma(g) \cong \mathrm{SL}_2(\mathbb{F}_q[t])F/Z$$

where $F = \mathbb{F}_q^\times \oplus 1$, $Z = \mathbb{F}_q^\times I_2$

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Some supposed expanders are actually disconnected

Components of D_g and $\widetilde{D}_g$ for $g(t) = t^n$

q	2												
n	2	3	4	5	6	7	8	9	10	11	12	13	14
C	1	2^2	2^3	2^5	2^6	2^8	2^9	2^{11}	2^{12}	2^{14}	2^{15}	2^{17}	2^{18}
$\widetilde{C}$	2^1	2^3	2^4	2^6	2^7	2^{10}	2^{11}	2^{13}	2^{14}	2^{17}	2^{18}	2^{20}	2^{21}
q	2												
n	15	16	17	18	19	20	21	22	23	24	25	26	
C	2^{20}	2^{21}	2^{23}	2^{24}	2^{26}	2^{27}	2^{29}	2^{30}	2^{32}	2^{33}	2^{35}	2^{36}	
$\widetilde{C}$	2^{24}	2^{25}	2^{27}	2^{28}	2^{31}	2^{32}	2^{34}	2^{35}	2^{38}	2^{39}	2^{41}	2^{42}	
q	4												
n	2	3	4	5	6	7	8	9	10	11	12	13	
C	1	1	1	1	1	1	1	1	1	1	1	1	
$\widetilde{C}$	2^2	2^2	2^4	2^4	2^6	2^6	2^8	2^8	2^{10}	2^{10}	2^{12}	2^{12}	
q	8					16				32		64	
n	2	3	4	5	6	7	2	3	4	2	3	2	
C	1	1	1	1	1	1	1	1	1	1	1	1	
$\widetilde{C}$	2^3	2^3	2^6	2^6	2^9	2^9	2^4	2^4	2^8	2^5	2^5	2^6	

Future directions

- General Kac-Moody algebras and groups

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- Highest weight representations

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- Modular forms