Products of two involutions in finite simple groups

Johanna Rämö
j.ramo@qmul.ac.uk
Groups St Andrews 2009 in Bath

Queen Mary
University of London

A problem in the Kourovka notebook

- Describe the finite simple groups in which every element is a product of two involutions.
- Describe the finite simple groups in which every element is strongly real.

Strongly real elements

- Let G be a group.
- The element $g \in G$ is real if there is $x \in G$ s.t. $x^{-1}gx = g^{-1}$.
- The element $g \in G$ is strongly real if there is an involution $x \in G$ s.t. $x^{-1}gx = g^{-1}$.
- A group is called strongly real if all its elements are strongly real.

Strongly real elements

- If g is a product of two involutions, then it is strongly real.
  - if $g = xy$, then $x^{-1}gx = x^{-1}(xy)x = yx = g^{-1}$
- If $x^{-1}gx = g^{-1}$ for some involution x, then $g = (xg^{-1})x$.
- What if $g = x$?
- In finite simple groups every involution is in a Klein group.

Finite simple groups:

- product of two involutions $\iff$ strongly real

What is already known?

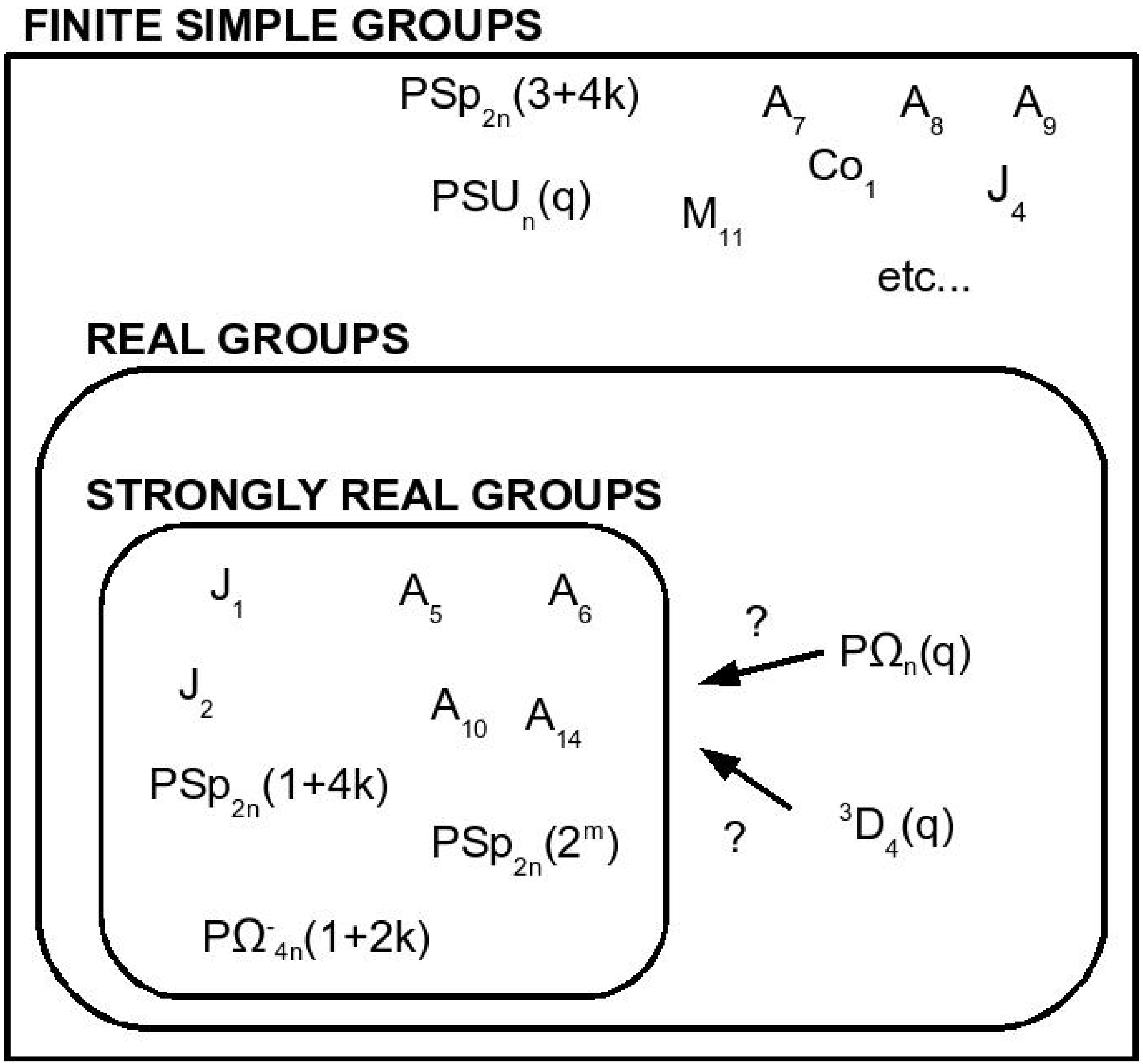
- Gow 1981, Ellers and Nolte 1982:
  $PSp_n(q)$ is strongly real if q is even.
- Bagiński 1987:
  Of the alternating groups only A_4, A_5, A_{10} and A_{14} are strongly real.
- Gow 1988:
  If $q \equiv 1 \pmod 4$, $PSp_n(q)$ is strongly real.
- Kolesnikov and Nuzhin 2005:
  Of the sporadic groups only J_1 and J_2 are strongly real.
- Galt 2009:
  If q is odd, $P\Omega^{-}_{4n}(q)$ is stronly real.

What is already known?

- Tiep and Zalesski 2005:
  Described the finite simple groups in which every element is real.
- A group cannot be strongly real if it is not real

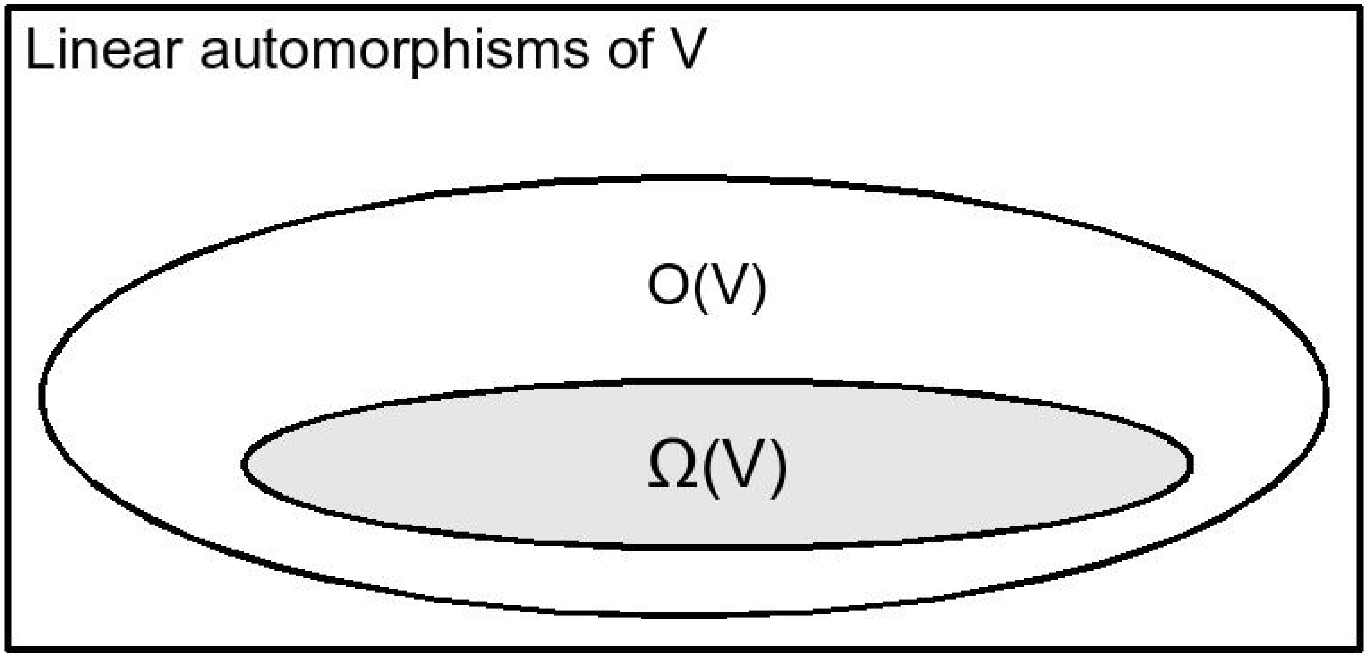
What is left?

- ${}^3D_4(q)$
- $P\Omega_n^{\varepsilon}(q)$ in some cases
  - Today we will talk about $P\Omega_n^{\varepsilon}(q)$, where q is even


FINITE SIMPLE GROUPS
$PSp_{2n}(3+4k)$
A_7
A_8
A_9
Co_1
$PSU_n(q)$
M_{11}
J_4
etc...
REAL GROUPS
STRONGLY REAL GROUPS
J_1
A_5
A_6
J_2
A_{10}
A_{14}
$PSp_{2n}(1+4k)$
$PSp_{2n}(2^m)$
$P\Omega^-_{4n}(1+2k)$
?
$P\Omega_n(q)$
?
$^3D_4(q)$

What does $P\Omega^{\varepsilon}_n(q)$ look like?

- Let V be an n-dimensional vector space over a field F of order q.
- Let $Q : V \to F$ be a quadratic form.
- The orthogonal group $O(V) = O^{\varepsilon}_n(q)$ consists of the linear automorphisms that preserve Q:
  - $O(V) = \{L : V \to V \mid Q(L(v)) = Q(v)$ for all $v \in V\}$
- ε is the type of Q
- $\Omega(V) \leq O(V)$.
- $P\Omega(V) = \Omega(V)/\{\text{scalars}\}$


Linear automorphisms of V
O(V)
Ω(V)

What does $P\Omega_n^\varepsilon(q)$ look like when q is even?

- If n is odd, then $O_n(q) \cong Sp_{n-1}(q)$.
- We can assume that n is even.
- $\Omega_n^\varepsilon(q) = \{L \in O_n^\varepsilon(q) \mid \text{Rank}(L + I_n) \text{ is even}\}$
- $P\Omega_n^\varepsilon(q) = \Omega_n^\varepsilon(q)$

Theorem (JR): If n and q are even, then the finite simple group $\Omega_n^{\varepsilon}(q)$ is strongly real if and only if $4 \mid n$.

- In other words, $\Omega_n^{\varepsilon}(q)$ is strongly real if and only if it is real.

Proof of the theorem:

- In $O_n^{\varepsilon}(q)$ all the elements are strongly real.
- This is proved by decomposing the vector space into a direct sum of orthogonal subspaces.
- In these subspaces it it easy to find the inverting involutions.
- We can assume that $4 \mid n$ because otherwise $\Omega_n^{\varepsilon}(q)$ is not real.
- For every element of $O_n^{\varepsilon}(q)$ the inverting involution can be chosen in such a way that it is in $\Omega_n^{\varepsilon}(q)$.

What is left?

- ${}^3D_4(q)$
- $P\Omega_n^+(q)$, where q is odd
- $P\Omega_n(q)$, where q and n are odd