

GENERALIZATIONS OF THE SYLOW THEOREM

Danila O. Revin¹

¹Sobolev Institute of Mathematics, Novosibirsk, Russia

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In 1872, the Norwegian mathematician L. Sylow proved the following outstanding theorem.

Theorem (L. Sylow)

Let G be a finite group, $|G| = p^\alpha m$, where p is a prime and $(p, m) = 1$. Then

- 1 (E-theorem) G possesses a subgroup of order p^α (the so-called Sylow p -subgroup);
- 2 (C-theorem) every two Sylow p -subgroups are conjugate;
- 3 (D-theorem) every p -subgroup of G is included in a Sylow p -subgroup of G .

A natural generalization of the concept of Sylow p -subgroups is the notion of π -Hall subgroups. Recall the definition.

By π we always denote a set of primes,

π' is its complement in the set of all primes.

By $\pi(n)$ we denote the set of prime divisors of an integer n .

An integer n is called a π -number, if $\pi(n) \subseteq \pi$.

For a finite group G we set $\pi(G) = \pi(|G|)$.

G is a π -group if $|G|$ is a π -number.

A subgroup H of G is called a π -Hall subgroup if $\pi(H) \subseteq \pi$ and $\pi(|G : H|) \subseteq \pi'$. The set of all π -Hall subgroups of G is denoted by $\text{Hall}_\pi(G)$ (note that this set may be empty).

So, if $\pi = \{p\}$ then every π -Hall subgroup of G is a Sylow p -subgroup and vice versa.

Following P. Hall, we say that a finite group G *satisfies*

E_π if $\text{Hall}_\pi(G) \neq \emptyset$ (i.e., there exists a π -Hall subgroup in G);

C_π if G satisfies E_π and every two π -Hall subgroups of G are conjugate;

D_π if G satisfies C_π and every π -subgroup of G is included in a π -Hall subgroup.

A group G satisfying E_π (resp. C_π , D_π) is said to be an E_π - (resp. C_π -, D_π -) *group*.

Given set of primes π , we also denote by E_π , C_π , and D_π the classes of all finite E_π -, C_π -, and D_π - groups, respectively.

Thus $G \in D_\pi$ if the complete analogue of Sylow's theorem holds for the π -Hall subgroups of G , while $G \in C_\pi$ and $G \in E_\pi$ if weaker analogues of Sylow's theorem hold.

Theorem (P.Hall, 1928)

If a finite group G is soluble then $G \in D_\pi$.

In contrast with Sylow's and Hall's theorems, there exists a set of primes π and a finite group G such that $\text{Hall}_\pi(G) = \emptyset$. There are examples showing that, in general, $E_\pi \neq C_\pi$ and $C_\pi \neq D_\pi$.

Alt_5 does not possess a subgroup of order 15, hence $\text{Alt}_5 \notin E_{\{3,5\}}$.

There are two classes of conjugate $\{2, 3\}$ -Hall subgroups of $\text{GL}_3(2)$: the stabilizers of lines and planes, respectively. So $\text{GL}_3(2) \in E_{\{2,3\}} \setminus C_{\{2,3\}}$.

Every subgroup of order 12 of Alt_5 is a point stabilizer, and all point stabilizers are conjugate. On the other hand, Alt_5 includes a $\{2, 3\}$ -subgroup $\langle (123), (12)(45) \rangle \simeq \text{Sym}_3$ which acts without fixed points. Therefore, $\text{Alt}_5 \in C_{\{2,3\}} \setminus D_{\{2,3\}}$.

Later P.Hall and, independently, S.A.Chunikhin proved the converse statement of Hall's theorem. Their results can be summarized in

Theorem (P.Hall and S.A.Chunikhin)

Let G be a finite group. The following statements are equivalent:

- 1 G is soluble;
- 2 $G \in D_\pi$ for every set π of primes;
- 3 $G \in E_{p'}$ for every prime p .

If we fix π , then the classes E_π , C_π , D_π can be wider than the class of solvable finite groups. It is clear, for example, that each π - or π' -group satisfies D_π .

In this talk we consider the following

General Problem

Given set π of primes and a finite group G , does G satisfy E_π , C_π or D_π ?

Let $\Phi \in \{E_\pi, C_\pi, D_\pi\}$. In order to solve this problem it is natural to ask whether a class Φ is closed under taking normal subgroups, homomorphic images and extensions?

Φ	E_π	C_π	D_π
normal subgroups	Yes	No	Yes (mod CFSG)
homomorphisms	Yes	Yes (mod CFSG)	Yes
extensions	No	Yes	Yes (mod CFSG)

Proposition

If $A \trianglelefteq G$ and $H \in \text{Hall}_\pi(G)$ then $HA/A \in \text{Hall}_\pi(G/A)$ and $H \cap A \in \text{Hall}_\pi(A)$.

As a consequence the following statements hold:

Proposition

If $A \trianglelefteq G$ and $G \in E_\pi$ then $G/A \in E_\pi$ and $A \in E_\pi$.

Proposition

If $G \in E_\pi$, then every composition factor of G satisfies E_π .

Problem

Describe the Hall subgroups in the finite simple groups.

This problem is solved by different authors. Evgeny Vdovin will tell about it in the next talk.

We need the following statement.

Extension Lemma

Let $A \trianglelefteq G$, $\pi(G/A) \subseteq \pi$, and $M \in \text{Hall}_\pi(A)$. Then there exists $H \in \text{Hall}_\pi(G)$ with $H \cap A = M$ if and only if G , acting by conjugation, leaves invariant the conjugacy class $M^A = \{M^a \mid a \in A\}$.

Let $\pi = \{2, 3\}$ and let $G = \text{GL}_3(2) = \text{SL}_3(2)$ be a group of order $168 = 2^3 \cdot 3 \cdot 7$. Then G has exactly two classes of conjugate π -Hall subgroups with the representatives

$$\left(\begin{array}{|c|c|} \hline \text{GL}_2(2) & * \\ \hline 0 & 1 \\ \hline \end{array} \right) \text{ and } \left(\begin{array}{|c|c|} \hline 1 & * \\ \hline 0 & \text{GL}_2(2) \\ \hline \end{array} \right).$$

The first one consists of the line stabilizers in the natural representation of G , and the second one consists of the plane stabilizers. The map $\iota : x \in G \mapsto (x^t)^{-1}$ is an automorphism of G of order 2. It interchanges the classes of π -Hall subgroups, hence by the Extension Lemma the group $\widehat{G} = G : \langle \iota \rangle$ does not possess a π -Hall subgroup.

This example shows that E_π is not closed under extension.

In 1986 F.Gross obtained a sufficient condition for a finite group G to satisfy E_π in terms of so-called induced automorphisms' groups of the composition factors of G .

Definition

Let A, B, H be subgroups of G such that $B \trianglelefteq A$. Then $N_H(A/B) = N_H(A) \cap N_H(B)$ is the *normalizer* of A/B in H . If $x \in N_H(A/B)$ then x induces an automorphism of A/B by $Ba \mapsto Bx^{-1}ax$. Thus there exists a homomorphism $N_H(A/B) \rightarrow \text{Aut}(A/B)$. The image of $N_H(A/B)$ under this homomorphism is denoted by $\text{Aut}_H(A/B)$ and is called a *group of H -induced automorphisms* of A/B . In the case $B = 1$, we write $\text{Aut}_H(A) = \text{Aut}_H(A/B)$.

Theorem (F. Gross, 1986, mod CFSG)

Let $1 = G_0 < G_1 < \dots < G_n = G$ be a composition series for a finite group G which is a refinement of a chief series for G . If $\text{Aut}_G(G_i/G_{i-1}) \in E_\pi$ for all $i = 1, \dots, n$, then $G \in E_\pi$.

By using the classification of finite simple groups and the description of π -Hall subgroups in such groups the following theorem is obtained.

Theorem (E. Vdovin and D. R., mod CFSG)

Let $1 = G_0 < G_1 < \dots < G_n = G$ be a composition series for a finite group G . If, for some i , $\text{Aut}_G(G_i/G_{i-1}) \notin E_\pi$, then $G \notin E_\pi$.

Corollary (E_π -criterion, mod CFSG)

Let $1 = G_0 < G_1 < \dots < G_n = G$ be a composition series for a finite group G which is a refinement of a chief series for G . Then $G \in E_\pi$ iff $\text{Aut}_G(G_i/G_{i-1}) \in E_\pi$ for all $i = 1, \dots, n$.

Problem

Find all almost simple E_π -groups and their π -Hall subgroups.

Recall that a finite group G is said to be *almost simple*, if $S \simeq \text{Inn}(S) \leq G \leq \text{Aut}(S)$ for a nonabelian simple group S . Since π -Hall subgroups for every finite simple group S are known, in order to solve the problem, we need to understand, which π -Hall subgroups can be lifted to π -Hall subgroups of the corresponding almost simple group G . Moreover, since G/S is solvable, Hall's Theorem implies that we may assume G/S to be a π -group. Thus the Extension Lemma is applicable, and by studying the action of G on the set of classes of conjugate π -Hall subgroups we'll be able to solve this problem. As a particular case, we mention the following result.

Corollary (mod CFSG)

Suppose a set of primes π is chosen so that $2 \notin \pi$ or $3 \notin \pi$. Then $G \in E_\pi$ for a finite group G iff $S \in E_\pi$ for every composition factor S of G .

From the E_π -criterion another interesting statement follows. We have mentioned that if A is a normal subgroup of an E_π -group G , then there exist maps $\text{Hall}_\pi(G) \rightarrow \text{Hall}_\pi(G/A)$ and $\text{Hall}_\pi(G) \rightarrow \text{Hall}_\pi(A)$, given by $H \mapsto HA/A$ and $H \mapsto H \cap A$, respectively. The first map turns out to be surjective.

Corollary (mod CFSG)

Every π -Hall subgroup in a homomorphic image of an E_π -group G is the image of a π -Hall subgroup of G .

The map $H \mapsto H \cap A$, in general, is not surjective, as we will see below.

The previous corollary implies a statement which related to the study of C_π -groups.

Corollary (mod CFSG)

If $G \in C_\pi$ and $A \trianglelefteq G$ then $G/A \in C_\pi$.

Theorem (S.A.Chunikhin)

Let A be a normal subgroup of G . If both A and G/A satisfy C_π , then $G \in C_\pi$.

Let us consider the following example.

Suppose $\pi = \{2, 3\}$. Let $G = \text{GL}_5(2)$. Consider the automorphism $\iota : x \in G \mapsto (x^t)^{-1}$ and the natural semidirect product $\widehat{G} = G : \langle \iota \rangle$. One can show that G possesses π -Hall subgroups, and each such subgroup is the stabilizer in G of a series $\{0\} = V_0 < V_1 < V_2 < V_3 = V$, where V is the natural module for G , and $\dim V_k/V_{k-1} \in \{1, 2\}$ for each $k = 1, 2, 3$. Therefore, there exist exactly three classes of conjugate π -Hall subgroups of G with the representatives

$$H_1 = \left(\begin{array}{ccc} \boxed{\text{GL}_2(2)} & & * \\ & \boxed{1} & \\ 0 & & \boxed{\text{GL}_2(2)} \end{array} \right),$$

$$H_2 = \begin{pmatrix} \boxed{1} & & * \\ & \boxed{\text{GL}_2(2)} & \\ 0 & & \boxed{\text{GL}_2(2)} \end{pmatrix},$$

$$H_3 = \begin{pmatrix} \boxed{\text{GL}_2(2)} & & * \\ & \boxed{\text{GL}_2(2)} & \\ 0 & & \boxed{1} \end{pmatrix}.$$

For each π -Hall subgroup H of \widehat{G} , the intersection $H \cap G$ is conjugate with one of H_1, H_2, H_3 . The class H_1^G is ι -invariant, so by the Extension Lemma there exists a π -Hall subgroup H of \widehat{G} with $H \cap G = H_1$. Now ι interchanges H_2^G and H_3^G . Thus H_2 and H_3 are not included in π -Hall subgroups of \widehat{G} . Therefore, \widehat{G} has exactly one class of conjugate π -Hall subgroups, so $\widehat{G} \in C_\pi$, while its normal subgroup G does not satisfy C_π .

This example also shows that, for a normal subgroup A of an E_π -group G , the map $\text{Hall}_\pi(G) \rightarrow \text{Hall}_\pi(A)$, given by $H \mapsto H \cap A$, is not surjective in general.

The condition $2 \in \pi$ is essential in the example, since F. Gross in 1987 proved (mod CFSG) the following:

Theorem (F.Gross, 1987, mod CFSG)

If π is a set of primes and $2 \notin \pi$ then $E_\pi = C_\pi$.

However, it is possible to show that some normal subgroups of a C_π -group satisfy C_π . Moreover, a criterion of C_π is obtained.

Theorem (E. Vdovin and D. R., mod CFSG)

Let $G \in C_\pi$, $H \in \text{Hall}_\pi(G)$, and $A \trianglelefteq G$. Then $HA \in C_\pi$.

Corollary (C_π -criterion, mod CFSG)

Let $A \trianglelefteq G$. Then $G \in C_\pi$ if and only if $G/A \in C_\pi$ and, for every (some) intermediate subgroup $A \leq K \leq G$ such that $K/A \in \text{Hall}_\pi(G/A)$, we have $K \in C_\pi$.

Corollary (mod CFSG)

If $A \trianglelefteq G$ and $|G : A|$ is a π' -number, then $G \in C_\pi \Leftrightarrow A \in C_\pi$.

Problem

Describe all finite almost simple groups satisfying C_π .

We illustrate the importance of this problem.

Lemma

Suppose $G = HA$, where $H \in \text{Hall}_\pi(G)$ and $A \trianglelefteq G$, and $A = S_1 \times \dots \times S_k$ is a direct product of finite simple groups. Then $G \in C_\pi$ if and only if $\text{Aut}_G(S_i) \in C_\pi$ for each $i = 1, \dots, k$.

Assume that $G = G_0 > G_1 > \dots > G_n = 1$ is a chief series of G . Set $H_1 = G = G_0$ and suppose that, for some $i \leq n$, subgroup H_i satisfies $G_{i-1} \leq H_i$ and $H_i/G_{i-1} \in \text{Hall}_\pi(G/G_{i-1})$. We have $G_{i-1}/G_i = S_1^i \times \dots \times S_{k_i}^i$, where $S_1^i, \dots, S_{k_i}^i$ are simple groups. We check whether $\text{Aut}_{H_i}(S_1^i) \in C_\pi, \dots, \text{Aut}_{H_i}(S_{k_i}^i) \in C_\pi$. If so, then the above lemma implies $H_i/G_i \in C_\pi$, and we denote the complete pre-image of a π -Hall subgroup of H_i/G_i by H_{i+1} . Otherwise, $G \notin C_\pi$ by the C_π -criterion and we stop the process. By the C_π -criterion, $G \in C_\pi$ iff H_{n+1} can be constructed.

Thus, the question, whether or not a given group G satisfies C_π is reduced to the same question about almost simple groups. Like in case of E_π -property, this problem is reduced to the investigation of the action of $G \leq \text{Aut}(S)$ on the set of classes of conjugate π -Hall subgroups of S , where S is a nonabelian finite simple group, and $S \simeq \text{Inn}(S) \leq G \leq \text{Aut}(S)$. In view of the C_π -criterion, we may also assume that G/S is a π -group. Now $G \in C_\pi$ if and only if S possesses exactly one G -invariant class of conjugate π -Hall subgroups. Since, for each finite simple group S , the classes of conjugate π -Hall subgroups are known, a solution to problem about almost simple C_π -groups will be obtained soon.

The theory of D_π -groups is the most complete at the moment, and the main results can be formulated in quite a simple way. It is not difficult to demonstrate that D_π is closed under taking homomorphic images.

Theorem (P.Hall, Theorem D5, 1956)

Let A be a normal subgroup of G . If $A, G/A \in D_\pi$, the π -Hall subgroups of A are nilpotent and the π -Hall subgroups of G/A are solvable then $G \in D_\pi$.

H.Wielandt, in the survey talk given at the XIII International mathematical congress in Edinburgh in 1958, proposed the following:

Problem, H. Wielandt, 1958, Kourovka Notebook, Problems 3.62 and 13.33

Let A be a normal subgroup of G .

- 1 Does G satisfy D_π if A and G/A both satisfy D_π ?
- 2 Does A satisfy D_π if G satisfies D_π ?

Theorem (E. Vdovin, V. Mazurov and D. R., 2006, mod CFSG)

Let A be a normal subgroup of G . Then $G \in D_\pi$ iff $A, G/A \in D_\pi$.

Thus, a finite group G satisfies D_π if and only if all composition factors of G satisfy D_π .

Furthermore, in several papers by F.Gross, E.Vdovin and D.R., all simple D_π -groups are described for every set π .

So, the question whether or not a given finite group G with known composition factors satisfies D_π become an arithmetical question.

Arithmetical criterion

Obtain an arithmetic description of finite almost simple E_π -groups (C_π -groups).

As we already mentioned this Problem remains open, although it is close to be solved.

Let G be a finite group. Consider classes V_π and W_π . We write $G \in W_\pi$ if, for every $H \leq G$, we have $H \in D_\pi$. We write $G \in V_\pi$ if $G \in E_\pi$ and $H \in E_\pi \Rightarrow H \in D_\pi$ for every $H \leq G$. It is clear that $W_\pi \subseteq V_\pi \subseteq D_\pi$. Moreover, $G \in V_\pi$ (resp. $G \in W_\pi$) iff $S \in V_\pi$ (resp. $S \in W_\pi$) for every composition factor S of G .

(H. Wielandt, Santa Cruz, 1979)

For every set of primes π , find all finite simple D_π -groups with one of the following properties:

- 1 all subgroups satisfy D_π ;
- 2 all E_π -subgroups satisfy D_π .

It is evident that if $A \leq G$ contains a π -Hall subgroup of $G \in E_\pi$ then $A \in E_\pi$.

C_π and D_π conjectures

Let $A \leq G$ contain a π -Hall subgroup of $G \in C_\pi$ (resp. $G \in D_\pi$). Then $A \in C_\pi$ (resp. $A \in D_\pi$).

Recall that a subgroup H of G is called *pronormal* if, for every $g \in G$, the subgroups H and H^g are conjugate in $\langle H, H^g \rangle$. We say that a subgroup H of G is *strongly pronormal* if K^g is conjugate to a subgroup of H in $\langle H, K^g \rangle$ for every $K \leq H$ and $g \in G$. Thus an equivalent version of the above problem can be formulated in the following way:

C_π and D_π conjectures

The π -Hall subgroups in C_π -groups (resp. in D_π -groups) are pronormal (resp. strongly pronormal).

Problem

Are the π -Hall subgroups in finite simple groups (strongly) pronormal?

This is not true for nonsimple groups.

Suppose $\pi = \{2, 3\}$, $X = \text{GL}_3(2)$. As we already noted, X possesses exactly two classes of conjugate π -Hall subgroups. Let Y be a cyclic subgroup of order 5 in Sym_5 , $G = X \wr Y$ and let

$$H \simeq \underbrace{X \times \dots \times X}_{5 \text{ times}}$$

be the base of this wreath product. Clearly, H has 2^5 classes of conjugate π -Hall subgroups. Since H is normal in G and $|G : H| = 5$ is a π' -number, $\text{Hall}_\pi(G) = \text{Hall}_\pi(H)$. Now Y acts on the set of classes of conjugate π -Hall subgroups of H . By using this fact it is easy to check that G has 8 classes of conjugate π -Hall subgroups. So there exist π -Hall subgroups of G which are conjugate in G , but are not conjugate even in their common normal closure H , whence this subgroups are not pronormal.

There are several “arithmetical” problems.

Denote by \mathfrak{G} the class of all finite groups, and by B_π the class of groups such that the order of each composition factor is either a π -number, or is divisible by at most one prime in π .

Problem

Find sets π of primes such that one or more inclusions in

$$B_\pi \subseteq D_\pi \subseteq C_\pi \subseteq E_\pi \subseteq \mathfrak{G}$$

are identities.

Trivial examples of

$$B_\pi = D_\pi = C_\pi = E_\pi = \mathfrak{G}$$

arises from the cases, when π is the empty set, the set of all primes, and a one- element set.

The classification of simple D_π -groups implies that if $2, 3 \in \pi$, then $B_\pi = D_\pi$.

Suppose $2, 3 \in \pi$ and $\pi' \neq \emptyset$. Set $p = \min \pi'$. Consider $G = \text{Sym}_p$. Each π -Hall subgroup of G is a point stabilizer in the natural permutation representation and isomorphic to Sym_{p-1} . So $G \in C_\pi$. Consider a π -subgroup $K = \langle (1, 2, \dots, p-2)(p-1, p) \rangle$ of G . K acts without fixed points, hence is not included in a π -Hall subgroup of G .

Proposition

Let π be a set of primes and $2, 3 \in \pi$, and $\pi' \neq \emptyset$. Then $D_\pi \neq C_\pi$.

P.Hall's conjecture, 1956

If $2 \notin \pi$ then $E_\pi = C_\pi = D_\pi$.

Theorem (Z.Arad and M.Ward, 1982, mod CFSG)

If $\pi = 2'$ then $D_\pi = C_\pi = E_\pi$.

Theorem (F.Gross, 1984)

$E_\pi \neq D_\pi$ for every finite set π of odd primes with $|\pi| \geq 2$.

Theorem (F.Gross, 1987, mod CFSG)

If π is a set of primes and $2 \notin \pi$ then $E_\pi = C_\pi$.

Theorem (D.Revin, mod CFSG)

Given a real number x , denote by π_x the set of primes p such that $p > x$. If $x \geq 7$ then $E_{\pi_x} = C_{\pi_x} = D_{\pi_x}$.

Conjecture

There exists a continuum number of sets π of primes such that $E_\pi = C_\pi = D_\pi$.

It is a weaker version of the above-mentioned Hall's conjecture.

Thank you for attention!