Languages in Group Theory

Recent Past and Open Problems

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Groups St Andrews 2009 in Bath

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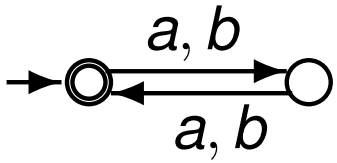

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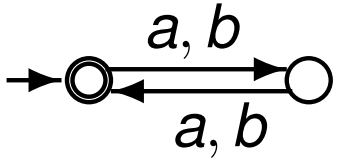

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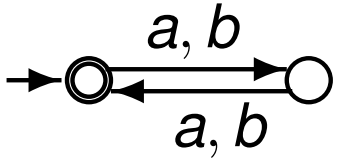

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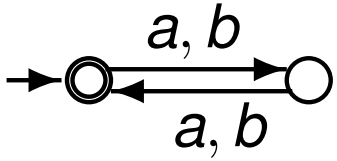

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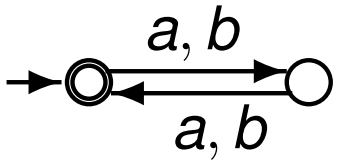

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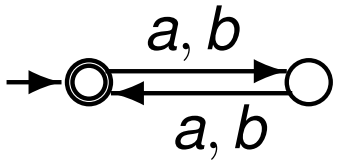

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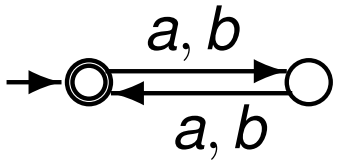

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  $CP_X(G) = \{ (u, v) \in A^{*2} \mid \exists g \in G : \ u^g = v \}$

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Being in state σ with ω at the top of the stack, when a is read from the input tape, the machine chooses (randomly) a transition with left hand side (σ, ω, a), say τ, and then replaces ω by μ and moves into state σ'.

The automaton accepts the input word, if the stack is empty at the end of some computation.

Prototypical Pushdown Automaton

Example

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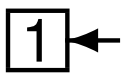

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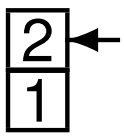

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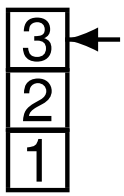

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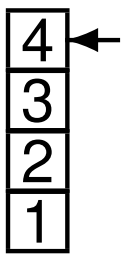

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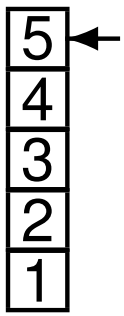

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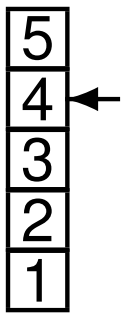

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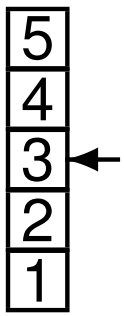

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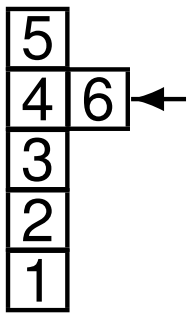

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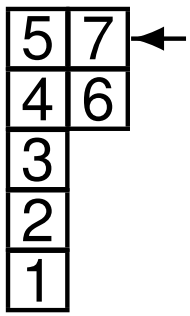

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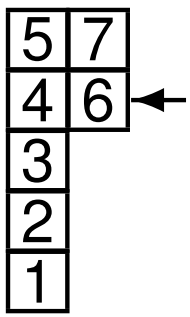

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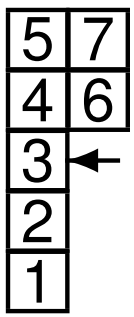

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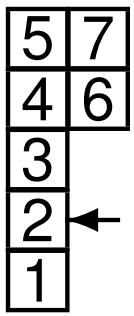

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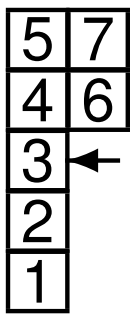

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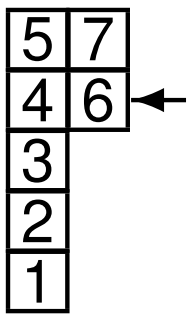

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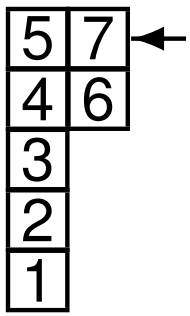

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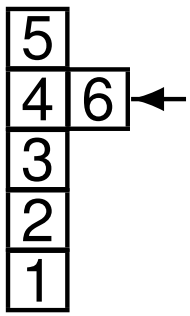

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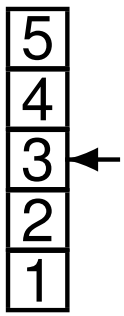

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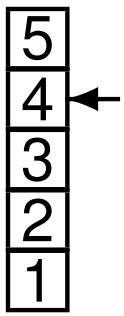

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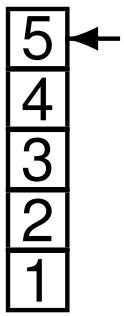

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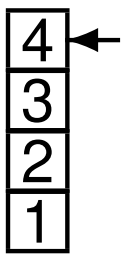

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Problem

Is a group with indexed word problem virtually free?

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1. $WP(\langle a, b \mid ab = ba^p \rangle)$ *is* 5*-tape tidy real-time.*
2. $WP(F_2 \times F_2)$ *is* 2*-tape but not* 1*-tape real-time.*

Problem

Show that $WP({F_2}^n)$ *is* n*-tape but not* $(n-1)$*-tape real-time.*

Context-Free Coword Problems

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Theorem (Lehnert-Schweitzer 2006)

All Higman-Thompson groups $G_{n,r}$ have context-free coword problem.

Indexed Coword Problems

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Problem

- *Show that* $coWP((\mathbb{Z} \times \mathbb{Z}) * C_2)$ *is not context-free.*
- *Show that the coword problem of 'the' Grigorchuk group is not context-free.*

Context-Free Conjugacy Problem

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Theorem (Holt-Rees-R 2009)

	sync. context-free	*async. context-free*
$CP(G)$	*virtually cyclic*	*virtually cyclic*
$ICP(G)$	*virtually cyclic*	*virtually free*

The End