Engel Groups
(a survey)

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Definition. Let G be a group and $a \in G$.

(a) We say that G is an *Engel* group if for each pair $(x, y) \in G \times G$ there exists an integer $n = n(x, y)$ such that

$$[y,_n x] = 1.$$

(b) We say that a is a *left Engel* element if for each $x \in G$ there exists an integer $n = n(x)$ such that

$$[x,_n a] = 1.$$

(c) We say that G is a *right Engel* element if for each $x \in G$ there exists an integer $n = n(x)$ such that

$$[a,_n x] = 1.$$

In (a) we call the number $\sup\{n(x, y) : (x, y) \in G \times G\}$, the Engel degree of G.

If in (a) we have $n(x, y) \leq m$ for all $(x, y) \in G \times G$ then we say that G is an m-Engel group and if in (b)[(c)] we have that $n(x) \leq m$ for all $x \in G$ then we say that a is a *left* [*right*] m-Engel element.

Observation

(a) G locally nilpotent $\Rightarrow$ G is an Engel group.
(b) a a left Engel element $\Leftarrow$ a is in the locally nilpotent radical.
(c) a a right Engel element $\Leftarrow$ a is in the hyper centre.

... Although some useful facts have been brought to light about Engel groups by K. W. Gruenberg and also by the attempts on the Burnside problem, the word problem for E_n remains unsolved for $n > 2$. Problems such as these still seem to present a formidable challenge to the ingenuity of algebraists. In spite of, or perhaps because of, their relatively concrete and particular character, they appear, to me at least, to offer an amiable alternative to the ever popular pursuit of abstractions. (P. Hall, 1958)

1) Origin. 2-Engel groups.

Observation (Burnside,1901)

$x^3 = 1 \;\Rightarrow xx^y = x^y x$ (or equivalently $[y, x, x] = 1$)

Structure

$$\begin{aligned} [x, y, z] &= [y, z, x] \\ [x, y, z]^3 &= 1 \qquad \text{(Burnside, 1902)} \\ [x, y, z, t] &= 1 \qquad \text{(Hopkins, 1929)} \end{aligned}$$

Problem 1. (a) Let G be a group of which every element commutes with all its endomorphic images. Is G nilpotent of class at most 2?

(b) Does there exist a finite 2-Engel 3-group of class three such that $\mathrm{Aut}\, G = \mathrm{Aut}_c G \cdot \mathrm{Inn}\, G$ where $\mathrm{Aut}_c G$ is the group of central automorphisms of G?

2) Zorn's Theorem and some generalisations

Theorem 1 (Zorn, 1936) *Every finite Engel group is nilpotent*

Theorem 2 (Gruenberg, 1953) *Every finitely generated solvable Engel group is nilpotent*

Theorem 3 (Baer, 1957) *Every Engel group satisfying max is nilpotent*

Theorem 4 (Suprenenko and Garščuk, 1962) *Every linear Engel group is nilpotent*

The analogs to the Burnside problems

b1) The general Burnside problem. Is every finitely generated periodic group finite?

b2) The Burnside problem. Let n be a given positive integer. Is every finitely generated group of exponent n finite?

b3) The restricted Burnside problem. Let r and n be given positive integers. Is there a largest finite r-generator group of exponent n?

e1) The general local nilpotence problem. Is every finitely generated Engel group nilpotent?

e2) The local nilpotence problem. Let n be a given positive integer. Is every finitely generated n-Engel group nilpotent?

e3) The restricted local nilpotence problem Let r and n be given positive integers. Is there a largest nilpotent r-generator n-Engel group?

Two results on Lie rings

Theorem 1 (Zel'manov, 1991) *Let* $L = \langle a_1, \dots, a_r \rangle$ *be a finitely generated Lie ring and suppose that there exist positive integers* s, t *such that*

$$\sum_{\sigma \in Sym(s)} x x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(s)} = 0 \qquad (i)$$

$$xy^t = 0 \qquad (ii)$$

for all $x, x_1, \dots, x_s \in L$ *and all Lie products* y *of the generators* $a_1, a_2, \dots, a_r$.

Then L *is nilpotent.*

Theorem 2 (Zel'manov, 1987) *Any torsion free* n*-Engel Lie ring is nilpotent.*

2) Zorn's Theorem and some generalisations

Theorem 1 (Zorn, 1936) *Every finite Engel group is nilpotent*

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Theorem 3 (Baer, 1957) *Every Engel group satisfying max is nilpotent*

Theorem 4 (Suprenenko and Garščuk, 1962) *Every linear Engel group is nilpotent*

Theorem 5 (Wilson, 1991) *Every finitely generated residually nilpotent n-Engel group is nilpotent.*

Problem 2. Is every finitely generated n-Engel group nilpotent?

Theorem 6 (Medvedev, 2003)*Every compact Engel group is locally nilpotent*

Theorem 7 (Kim and Rhemtulla, 1991)*Every orderable n-Engel group is nilpotent*

Problem 3. Is every right orderable n-Engel group nilpotent?

Remark. True if $n = 4$ (Longobardi and Maj, 1997)

Theorem 8 (H. Smith, 2002)*Every n-Engel group with all subgroups subnormal is nilpotent.*

3) Structure of n-Engel groups.

Theorem 1 *Let G be a finitely generated n-Engel group that is non-nilpotent. There exists a finitely generated section S of G that is simple non-abelian.*

Locally nilpotent n-Engel groups

Theorem 2 (Burns and Medvedev, 1998) *There exist positive integers $m = m(n)$ and $r = r(n)$ such that for any locally nilpotent n-Engel group we have*

$$\gamma_{m+1}(G)^r = \{1\}.$$

Theorem 3 (T, Crosby, 2010) *There exist positive integers $m = m(n)$ and $r = r(n)$ such that any locally nilpotent n-Engel group satisfies the law*

$$[x^r, x_1, \ldots, x_m] = 1.$$

Theorem 4 (Abdollahi and T, 2002) *There exists a positive integer $s = s(n)$ such that every powerful n-Engel p-group is nilpotent of class at most s.*

Theorem 5 (Abdollahi and T, 2002) *Let p a prime and let $r = r(p, n)$ be the integer satisfying $p^{r-1} < n \leq p^r$. Let G be a locally finite n-Engel p-group.*

(a) If p is odd, then G^{p^r} is nilpotent of n-bounded class.
(b) If $p = 2$ then $(G^{2^r})^2$ is nilpotent of n-bounded class.

Some structure questions for n-Engel groups

$$G \text{ nilpotent} \implies G \text{ solvable} \implies G \text{ locally nilpotent.}$$

Question 1. Which primes need to be excluded to have nilpotence?
Question 2. Which primes need to be excluded to have solvability?

Theorem 6 (Gruenberg, 1961) *Let G be a solvable n-Engel group with derived length d. If G has no elements of prime order $p < n$ then G is nilpotent of class at most $(n+1)^{d-1}$.*

Question 3. When is a locally nilpotent n-Engel group a Fitting group?

Problem 4. Let G be an n-Engel p-group where $p > n$. Is G a Fitting group?

3-Engel groups

Theorem 1 (Heineken, 1961) *Every* 3*-Engel group is locally nilpotent.*

Theorem 2 (Heineken, 1961) *Let* G *be a* 3*-Engel group that is* $\{2, 5\}$*-free. Then* G *is nilpotent of class at most* 4.

Theorem 3 (L. Kappe, W. Kappe, 1972) *Let* G *be a* 3*-Engel group then* $\langle x \rangle^G$ *is nilpotent of class at most* 2 *for all* $x \in G$.

Theorem 4 (Gupta, 1972) *Every* 3*-Engel* 2*-group is solvable.*

Problem 5. To obtain a normal form theorem for the relatively free 3-Engel group of infinite countable rank.

The local nilpotence problem for 4-Engel groups

Let $G = \langle x_1, \ldots, x_r \rangle$ be a r-generated 4-Engel group. Suffices to show that G is nilpotent when $r = 3$.

1 $\langle x, x^y \rangle$ is nilpotent. (T, Longobardi and Maj, Vaughan-Lee) (1995-1997)

2 Every 4-Engel group that is either a 2-group or a 3-group is locally finite. Further more G^p is contained in the locally nilpotent radical for any p-group G. [T]

3. Every 4-Engel 5-group is locally finite. (Vaughan-Lee, 1997)

4. $\langle x, y \rangle$ is nilpotent (T, 2005)

5. 4-Engel groups are locally nilpotent (Havas, Vaughan-Lee, 2005)

4-Engel groups

Theorem 1 (Havas and Vaughan-Lee, 2005) *Every 4-Engel group is locally nilpotent.*

Theorem 2 (T, 1995) *Let G be a 4-Engel group that is $\{2, 3, 5\}$-free. Then G is nilpotent of class at most 7.*

Theorem 3 (Abdollahi and T, 2002) *Every 4-Engel 3-group is solvable*

Theorem 4 (T, 2003) *Every 4-Engel group is a Fitting group of Fitting degree at most 4. Furthermore if G is $\{2, 5\}$-free then the Fitting degree is a most 3*

Theorem 5 (Vaughan-Lee, 2007) *G is a 4-Engel group if and only if $\langle x\rangle^G$ is 3-Engel for all $x \in G$.*

Problem 6. Let G be a group. Do the right 4-Engel elements belong to the locally nilpotent radical of G? Do the left 3-Engel elements belong to the locally nilpotent radical of G?

Problem 7. Describe the structure of 5-Engel groups?

4) Generalisations of Engel groups

A. Generalised Burnside varieties.

A variety of groups $\mathcal{V}$ is said to be a *strong generalised Burnside variety* if it satisfies the following equivalent properties.

1) For each positive integer r the class of all nilpotent r-generator groups in $\mathcal{V}$ is r-bounded.
2) Every finitely generated group G in $\mathcal{V}$ that is residually nilpotent is nilpotent.
3) The locally nilpotent groups in $\mathcal{V}$ form a subvariety.

Theorem 1 (T, 2005) *Let $\mathcal{V}$ be a variety. The following are equivalent.*

(1) $\mathcal{V}$ is a strong generalised Burnside variety.
(2) The groups $C_p \, wr \; C$ and $C \, wr \, C_p$ do not belong to $\mathcal{V}$ for any prime p.

Theorem 2 (Endimioni, 2002) *Let $\mathcal{V}$ be a variety. The following are equivalent.*

(1) $\mathcal{V}$ is a strong generalised Burnside variety.
(2) There exist positive integers c, e such that all locally nilpotent groups in $\mathcal{V}$ are both in $\mathcal{N}_c\bar{\mathcal{B}}_e$ and $\bar{\mathcal{B}}_e\mathcal{N}_c$.

Theorem 3 (Zel'manov, 1993) *The variety $\overline{B}_p$ is finitely based for every prime p*

Proposition (Endimioni, written correspondence) *The following are equivalent*

(1) The variety $\overline{\mathcal{B}}_n$ of all locally nilpotent groups of exponent n is finitely based for all positive integers n.
(2) The variety $\overline{\mathcal{E}}_n$ of all locally nilpotent n-Engel groups is finitely based for all positive integers n.
(3) If a strong generalised Burnside variety $\mathcal{V}$ is finitely based then so is the variety $\overline{\mathcal{V}}$ consisting of all locally nilpotent groups in $\mathcal{V}$.

Problem 8. Is it true that for every strong generalised Burnside variety $\mathcal{V}$ that is finitely based, we have that the variety $\overline{\mathcal{V}}$ of all the locally nilpotent groups in $\mathcal{V}$ is also finitely based?

Remark. In particular if all n-Engel groups are locally nilpotent then the answer is yes.

B. Generalised Engel groups

Let G be any group. For $a, t \in G$, let $H = H(a,t) = \langle a \rangle^{\langle t \rangle}$ and

$$A(a,t) = H/[H,H].$$

Then $A(a,t)$ is an abelian section of G. Let $E(a,t)$ be the ring of all endomorphisms of $A(a,t)$. Notice that t induces an endomorphism on $A(a,t)$ by conjugation.

Definition. Let $I \trianglelefteq \mathbb{Z}[x]$. We say that G is an I-group if

$$a^{f(t)} = 0$$

in $A(a,t)$ for all $a, t \in G$ and for all $f \in I$.

For example any n-Engel group is an $\mathbb{Z}[x](x-1)^n$-group.

If G is any group then the set of polynomials f, such that $a^{f(t)} = 0$ in $A(a,t)$ for all $a, t \in G$, form an ideal $I(G)$. There is therefore a unique maximal ideal I such that G is an I-group. We say that two groups H and G are $\mathbb{Z}[x]$-equivalent if $I(H) = I(G)$.

Theorem 1 [T, 2005] *Let $f \in \mathbb{Z}[x]$ such that f is neither divisible by p nor f_p for all primes p. For each positive integer r there exists a positive integer $c(r,f)$ such that*

$$\gamma_{c(r,f)} = \{1\}$$

for any nilpotent r-generator $\mathbb{Z}[x]f$-group in G.

Theorem 2 [T] *Let $f \in \mathbb{Z}[x]$ such that f is neither divisible by p nor f_p for all primes p. There exist positive integers $c(f)$ and $e(f)$ such that*

$$[G^{e(f)},_c G] = (\gamma_{c(f)}(G))^{e(f)} = \{1\}$$

for any nilpotent $\mathbb{Z}[x]f$-group in G