On the linearity of HNN-extensions with abelian base groups

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Let $G = \langle\, t,\, K \mid t^{-1}At = B,\, \varphi\,\rangle$ be the HNN-extension with base group K a polycyclic-by-finite group and associated subgroups A and B of finite index in K. In *International Journal of Algebra and Computation Vol. 5, (1995) 719-724* a characterization for the $\mathbb{Z}$-linearity of G is given.

More precisely it is proved the

Theorem (R. T. V.): *G is $\mathbb{Z}$-linear if and only if it is subgroup separable.*

In case where the associated subgroups are of infinite index in the base group the situation is very different and the arguments development there are not applied even the base group is a finitely generated abelian group.

Here we obtain a characterization for the $\mathbb{Z}$-linearity of the HNN-extension $G = \langle\, t,\, K \mid t^{-1}At = B,\, \varphi\,\rangle$ with base group K a f. g. abelian group in the case where the associated subgroups are of infinite index in K.

Theorem A: *Let K be a f. g. abelian group and A, B proper isomorphic subgroups of K with $\varphi : A \longrightarrow B$ an isomorphism and $G = \langle\, t,\, K \mid t^{-1}At = B,\, \varphi\,\rangle$ be the corresponding HNN-extension. The group G is $\mathbb{Z}$-linear if and only if it is residually finite.*

Before describing the procedure for the proof of the Theorem A we define a subgroup of the group K (Here and in sequel K is a f. g. abelian group and A, B proper isomorphic subgroups of K) and we refer to a Theorem.

Let $D = \{\, x \in K \mid$ for each $\nu \in \mathbb{Z}$ there exists $\lambda = \lambda(\nu) \in \mathbb{N}$ such that $t^{-\nu}x^{\lambda}t^{\nu} \in K\,\}$, this group plays a central role in the sequel. It is isolated in the sense that if whenever $k^n \in D$ for $k \in K$ and $n > 0$, then $k \in D$. So it is a direct factor of K and contains the torsion part of K. We can see the subgroup D under the view of the Bass-Serre theory.

Let T the standard tree on which G acts, the subgroup D is the subgroup of K such that for every subtree T' of T, there is a positive integer n such that D^n stabilizes T' pointwise.

Let H be the "largest" subgroup of K such that $\varphi(H) = H$ (largest in the sense that if $L \leq L$ and $\varphi(L) = L$, then $L \leq H$). This subgroup is the largest subgroup of K that stabilizes the entire tree T pointwise.

Theorem (A.R.V.): *Let K be a f. g. abelian group, A, B proper subgroups of K and*
$\varphi : A \longrightarrow B$ an isomorphism. The following are equivalent:
(i) There exists a finitely generated abelian group X such that K is a subgroup of finite index in X and an automorphism $\bar{\varphi} \in Aut(X)$ with $\bar{\varphi}_{|A} = \varphi$.
(ii) There exists a subgroup $H \leq_f D$ such that $\varphi(H) = H$.
(iii) The corresponding HNN-extension
$G = \langle\, t,\, K \mid t^{-1}At = B,\, \varphi \,\rangle$ is residually finite.

see in [*Archiv Math. 50 (1988), 495-501*] and in [*Arch. Math., Vol. 53 (1989) 121-125*]

Let $D = 1$ and $A \cap B \neq 1$. We take a non trivial element $c_1 \in A \cap B$, apply the isomorphism φ and obtain the two side sequence
$\ldots, \varphi^{-2}(c_1), \varphi^{-1}(c_1), c_1, \varphi(c_1), \varphi^2(c_1), \ldots$.

Since $D = 1$, this sequence is finite, namely there are positive integers λ_1, μ_1 such that $\langle\, \varphi^{-\lambda_1}(c_1) \,\rangle \cap B = 1$ and $\langle\, \varphi^{\mu_1}(c_1) \,\rangle \cap A = 1$. It is easy to see that the subgroup
$S_1 = \langle\, \varphi^{-\lambda_1}(c_1), \ldots, \varphi^{-2}(c_1), \varphi^{-1}(c_1), c_1, \varphi(c_1), \varphi^2(c_1), \ldots, \varphi^{\mu_1}(c_1) \,\rangle$ is a free abelian group of rank $\lambda_1 + 1 + \mu_1$ and the cyclic permutation $j \longrightarrow j+1$ taken $mod(\lambda_1 + 1 + \mu_1)$ on the exponents of $\varphi^j(c_1)$ defines an automorphism, say φ_1 of S_1 of finite order. Moreover this automorphism coincides with φ in the sense that $\varphi_1 \mid_{A \cap S_1} = \varphi$.

Let $c_2 \in A \cap B$ such that $\langle\, c_2 \,\rangle \cap S_1 = 1$. We repeat the above process and construct a subgroup S_2 and an automorphism φ_2 of finite order such that
$\varphi_2 \mid_{A \cap S_2} = \varphi$.

Repeating the above procedure we construct the subgroups $S_1, S_2, \ldots, S_n$ and the automorphisms φ_i until we cover the free rank of $A \cap B$. The "lucky thing" is that we can prove that the automorphisms coincide in the intersections ($\varphi_{i_{|S_i \cap S_j}} = \varphi_{j_{|S_i \cap S_j}}$). This allows us to obtain the following

Proposition 1: *Let* $G = \langle\, t,\, K \mid t^{-1}At = B,\, \varphi \,\rangle$ *be the HNN-extension with base group* K *a f. g. abelian group, where* φ *is the isomorphism induced by* t*, Suppose that the subgroup* D *defined above is trivial. Then there exists a finite index subgroup* K_1 *of* K *in which* φ *induces an automorphism* φ_1 *of finite order.*

The next Proposition is critical for the linearity of these groups.

Proposition 2: *Let* $G = \langle\, t_1,\, t_2,\, \dots\,,\, K \mid t_i^{-1}at_i = \varphi(a),\, a \in A \,\rangle$ *be the multiple HNN-extension with base group* K *a f. g. free abelian group, where* A *is a subgroup of* K *and* φ *is an automorphism of* K *of finite order. The group* G *is* $\mathbb{Z}$*-linear.*

Sketch of the Proof: Since φ is an automorphism of K we construct an other HNN-extension, say $\bar{G}$, and embed G into $\bar{G}$. Because φ is of finite order there is an epimorphism from the group $\bar{G}$ onto a finite group. Let M be the kernel of this epimorphism. We prove that M is a right-angled Artin group. Consequently it is $\mathbb{Z}$-linear (for the linearity of right-angled Artin groups see in (Brown, K.S., *Buildings* Springer-Verlag, N.Y. 1989) and in [Michigan Math. J., 46 (1999), 251-259]. This implies the $\mathbb{Z}$-linearity of $\bar{G}$. Namely the $\mathbb{Z}$-linearity of G.

An easy combination of the previous Propositions gives the following proposition.

Proposition 3: *Let* $G = \langle\, t,\, K \mid t^{-1}At = B,\, \varphi \,\rangle$ *be the HNN-extension with base group* K *a f. g. abelian group, where* φ *is the isomorphism induced by* t*, Suppose that the subgroup* D *defined above is trivial, then the group* G *is* $\mathbb{Z}$*-linear.*

For the general case we need one more step.

Proposition 4: *Let* $G = \langle\, t,\, K \mid t^{-1}At = B,\, \varphi \,\rangle$ *be the HNN-extension with base group* K *a f. g. abelian group, where* φ *is the isomorphism induced by* t*. Suppose that the subgroup* D *defined above is finite, then the group* G *is* $\mathbb{Z}$*-linear.*

Sketch of the Proof: Since the group D is finite, the group G is residually finite (by the Theorem (A.R.V.)), therefore there exists a finite index normal subgroup N of G such that $N \cap D = 1$. Now for the torsion free group $K_1 = K \cap N$ and the corresponding subgroup $D_1 = \{\, x \in K_1 \mid \text{for each } \nu \in \mathbb{Z} \text{ there exists } \lambda = \lambda(\nu) \in \mathbb{N} \text{ such that } t^{-\nu}x^{\lambda}t^{\nu} \in K_1 \,\}$, which is trivial, we can apply the previous results to obtain a finite index subgroup in G, which (and therefore the group G) is $\mathbb{Z}$-linear.

Let now $G = \langle t, K \mid t^{-1}At = B, \varphi \rangle$ be the HNN-extension with base group K a f. g. abelian group, where φ is the isomorphism induced by t. Suppose that the subgroup D defined above is not finite, to reduce this general case in the case where D

is finite we use the subgroup H of G defined above as the largest subgroup of K with the property $\varphi(H) = H$ and the fact that G is residually finite if and only if H is of finite index in D (see the Theorem (A.R.V.)) and prove the

Theorem B: *Let $G = \langle t, K \mid t^{-1}At = B, \varphi \rangle$ be the HNN-extension with base group K a f. g. abelian group, where φ is the isomorphism induced by t. The group G is $\mathbb{Z}$-linear if and only if the quotient group G/H is $\mathbb{Z}$-linear.*

Sketch of the Proof: Suppose that G is linear then it is residually finite, therefore H is of finite index in D. Taking the quotient group G/H we have that it has an HNN-presentation with the corresponding D_H subgroup finite, therefore by the previous Proposition the group G/H is $\mathbb{Z}$-linear.

Conversely, suppose that G/H is linear, this implies that there is an homomorphism $\vartheta : G \longrightarrow R$ to a linear group R with $Ker\vartheta \leq H$. Also the linearity of G/H implies the residual finiteness of it. This gives rise to an homomorphism $\varrho : G \longrightarrow X \rtimes \langle \bar{\varphi} \rangle$, where X is a f. g. abelian group and $\bar{\varphi}$ an automorphism of it. The homomorphism ϱ is an embedding of K, so $Ker\vartheta \cap Ker\varrho = 1$. This implies the linearity of G.

As Corollary of the previous Theorem is derived our main result

Theorem A: *Let K be a f. g. abelian group and A, B proper isomorphic subgroups of K with $\varphi : A \longrightarrow B$ an isomorphism and $G = \langle t, K \mid t^{-1}At = B, \varphi \rangle$ be the corresponding HNN-extension. The group G is $\mathbb{Z}$-linear if and only if it is residually finite.*

Proof: Suppose that G is residually finite, then, as we have said previously, the subgroup H is of finite index in D. This means that the group G/H is linear (by Proposition 4). Therefore G is linear (by Theorem B).

Remarks

i) If we embody the two Theorems A and (A.R.V.) we have the

THEOREM: *Let K be a f. g. abelian group, A, B proper subgroups of K and*
$\varphi : A \longrightarrow B$ an isomorphism. The following are equivalent:
(i) There exists a finitely generated abelian group X such that K is a subgroup of finite index in X and an automorphism $\bar{\varphi} \in Aut(X)$ with $\bar{\varphi}_{|A} = \varphi$.
(ii) There exists a subgroup $H \leq_f D$ such that $\varphi(H) = H$.
(iii) The corresponding HNN-extension
$G = \langle\, t,\, K \mid t^{-1}At = B,\, \varphi \,\rangle$ is residually finite.
(iv) The group G is $\mathbb{Z}$-linear.

ii) In Proposition 4 (via the proof of the Proposition 1) we have proved that if the group D is finite then there exists a finite index subgroup K_1 of K and an automorphism φ_1 of K_1 of **finite** order induced by φ and the group $G = \langle\, t,\, K \mid t^{-1}At = B,\, \varphi \,\rangle$ is $\mathbb{Z}$-linear. But the converse is not true as we can see in the following simple example: Let $G = \langle\, t,\, a,\, b \mid t^{-1}at = a,\, [a,\, b] = 1 \,\rangle$, then G trivially satisfies all the requires of the THEOREM, but the corresponding group $D = \langle\, a \,\rangle$ is not finite.

iii) The subgroups A, B are assumed to be proper subgroups of the base group K. In the case where one of them is all the base group, then the HNN-extension $G = \langle\, t,\, K \mid t^{-1}Kt = B,\, \varphi \,\rangle$ is residually finite and, as a constructible solvable group, it is $\mathbb{Q}$-linear (see for example in [Strebel, R., *Finitely presented soluble groups*, Group Theory, essays for Philip Hall, Academic Press 1894]). BUT this group is not $\mathbb{Z}$-linear. This is concluded from the fact that the subgroup B is not closed in the profinite topology of G, on the other hand if G was $\mathbb{Z}$-linear, then all subgroups of K must be closed in the profinite topology of G. (for details see in [*Demonstratio Mathematica Vol. XXIX (1996), 43-52*] and in [*International Journal of Algebra and Computation Vol. 5, (1995) 719-724*])

iv)Our arguments are based heavily on the structure of the base group K as a $\mathbb{Z}$-module. So we are unable (for the moment) to extent these techniques in the case where the base group is a polycyclic-by-finite group (even a f. g. nilpotent group).