HALL SUBGROUPS IN FINITE SIMPLE GROUPS

Evgeny P. Vdovin[1]

[1]Sobolev Institute of Mathematics SB RAS

Groups St Andrews 2009

The term "group" always means a finite group. By π we always denote a set of primes, π' is its complement in the set of all primes. A rational integer n is called a π-number, if all its prime divisors are in π, by $\pi(n)$ we denote all prime divisors of a rational integer n. For a group G we set $\pi(G)$ to be equal to $\pi(|G|)$ and G is a π-*group* if $|G|$ is a π-number.

A subgroup H of G is called a π-*Hall subgroup* if $\pi(H) \subseteq \pi$ and $\pi(|G:H|) \subseteq \pi'$. A set of all π-Hall subgroups of G we denote by $\mathrm{Hall}_\pi(G)$ (note that this set may be empty).

According to P. Hall we say that G *satisfies* E_π (or briefly $G \in E_\pi$), if G possesses a π-Hall subgroup. If $G \in E_\pi$ and every two π-Hall subgroups are conjugate, then we say that G *satisfies* C_π ($G \in C_\pi$). If $G \in C_\pi$ and each π-subgroup of G is included in a π-Hall subgroup of G, then we say that G *satisfies* D_π ($G \in D_\pi$). The number of classes of conjugate π-Hall subgroups of G we denote by $k_\pi(G)$.

The term "group" always means a finite group. By π we always denote a set of primes, π' is its complement in the set of all primes. A rational integer n is called a π-number, if all its prime divisors are in π, by $\pi(n)$ we denote all prime divisors of a rational integer n. For a group G we set $\pi(G)$ to be equal to $\pi(|G|)$ and G is a π-*group* if $|G|$ is a π-number.

A subgroup H of G is called a π-*Hall subgroup* if $\pi(H) \subseteq \pi$ and $\pi(|G:H|) \subseteq \pi'$. A set of all π-Hall subgroups of G we denote by $\mathrm{Hall}_\pi(G)$ (note that this set may be empty).

According to P. Hall we say that G *satisfies* E_π (or briefly $G \in E_\pi$), if G possesses a π-Hall subgroup. If $G \in E_\pi$ and every two π-Hall subgroups are conjugate, then we say that G *satisfies* C_π ($G \in C_\pi$). If $G \in C_\pi$ and each π-subgroup of G is included in a π-Hall subgroup of G, then we say that G *satisfies* D_π ($G \in D_\pi$). The number of classes of conjugate π-Hall subgroups of G we denote by $k_\pi(G)$.

The term "group" always means a finite group. By π we always denote a set of primes, π' is its complement in the set of all primes. A rational integer n is called a π-number, if all its prime divisors are in π, by $\pi(n)$ we denote all prime divisors of a rational integer n. For a group G we set $\pi(G)$ to be equal to $\pi(|G|)$ and G is a π-*group* if $|G|$ is a π-number.

A subgroup H of G is called a π-*Hall subgroup* if $\pi(H) \subseteq \pi$ and $\pi(|G : H|) \subseteq \pi'$. A set of all π-Hall subgroups of G we denote by $\mathrm{Hall}_\pi(G)$ (note that this set may be empty).

According to P. Hall we say that G *satisfies* E_π (or briefly $G \in E_\pi$), if G possesses a π-Hall subgroup. If $G \in E_\pi$ and every two π-Hall subgroups are conjugate, then we say that G *satisfies* C_π ($G \in C_\pi$). If $G \in C_\pi$ and each π-subgroup of G is included in a π-Hall subgroup of G, then we say that G *satisfies* D_π ($G \in D_\pi$). The number of classes of conjugate π-Hall subgroups of G we denote by $k_\pi(G)$.

Elementary properties of Hall subgroups

1. If $A \trianglelefteq G$ and $H \in \mathrm{Hall}_\pi(G)$, then $HA/A \in \mathrm{Hall}_\pi(G/A)$ and $H \cap A \in \mathrm{Hall}_\pi(A)$.
2. Assume that G possesses a subnormal series $\{e\} = G_0 < G_1 < \ldots < G_{k-1} < G_k = G$ such that the order of each section either is a π-number, or is divisible by at most one prime from π. Then G satisfies D_π, i. e., G possesses a π-Hall subgroup H and each π-subgroup of G is conjugate to a subgroup of H. In such case we say that a π-Hall subgroup of G is standard, otherwise a π-Hall subgroup of G is called nonstandard.

Elementary properties of Hall subgroups

1. If $A \trianglelefteq G$ and $H \in \mathrm{Hall}_\pi(G)$, then $HA/A \in \mathrm{Hall}_\pi(G/A)$ and $H \cap A \in \mathrm{Hall}_\pi(A)$.
2. Assume that G possesses a subnormal series $\{e\} = G_0 < G_1 < \ldots < G_{k-1} < G_k = G$ such that the order of each section either is a π-number, or is divisible by at most one prime from π. Then G satisfies D_π, i. e., G possesses a π-Hall subgroup H and each π-subgroup of G is conjugate to a subgroup of H. In such case we say that a π-Hall subgroup of G is standard, otherwise a π-Hall subgroup of G is called nonstandard.

Extension Lemma

If $A \trianglelefteq G$, $\pi(G/A) \subseteq \pi$, and $M \in \mathrm{Hall}_\pi(A)$, then there exists $H \in \mathrm{Hall}_\pi(G)$ with $H \cap A = M$ if and only if G, acting by conjugation, leaves invariant the set $\{M^a \mid a \in A\}$.

Let $\pi = \{2, 3\}$, $G = \mathrm{GL}_3(2) = \mathrm{SL}_3(2)$ be a group of order $168 = 2^3 \cdot 3 \cdot 7$. Then G has exactly two classes of π-Hall subgroups with representatives

$$\begin{pmatrix} \mathrm{GL}_2(2) & * \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & * \\ 0 & \mathrm{GL}_2(2) \end{pmatrix}.$$

The map $\iota : x \in G \mapsto (x^t)^{-1}$ is an automorphism of order 2 of G. It interchanges classes of π-Hall subgroups, hence the group $\widehat{G} = G : \langle \iota \rangle$ does not possess a π-Hall subgroup.

Extension Lemma

If $A \trianglelefteq G$, $\pi(G/A) \subseteq \pi$, and $M \in \mathrm{Hall}_\pi(A)$, then there exists $H \in \mathrm{Hall}_\pi(G)$ with $H \cap A = M$ if and only if G, acting by conjugation, leaves invariant the set $\{M^a \mid a \in A\}$.

Let $\pi = \{2, 3\}$, $G = \mathrm{GL}_3(2) = \mathrm{SL}_3(2)$ be a group of order $168 = 2^3 \cdot 3 \cdot 7$. Then G has exactly two classes of π-Hall subgroups with representatives

$$\begin{pmatrix} \boxed{\mathrm{GL}_2(2)} & * \\ 0 & \boxed{1} \end{pmatrix} \text{ and } \begin{pmatrix} \boxed{1} & * \\ 0 & \boxed{\mathrm{GL}_2(2)} \end{pmatrix}.$$

The map $\iota : x \in G \mapsto (x^t)^{-1}$ is an automorphism of order 2 of G. It interchanges classes of π-Hall subgroups, hence the group $\widehat{G} = G : \langle \iota \rangle$ does not possess a π-Hall subgroup.

Extension Lemma

If $A \trianglelefteq G$, $\pi(G/A) \subseteq \pi$, and $M \in \mathrm{Hall}_\pi(A)$, then there exists $H \in \mathrm{Hall}_\pi(G)$ with $H \cap A = M$ if and only if G, acting by conjugation, leaves invariant the set $\{M^a \mid a \in A\}$.

Let $\pi = \{2, 3\}$, $G = \mathrm{GL}_3(2) = \mathrm{SL}_3(2)$ be a group of order $168 = 2^3 \cdot 3 \cdot 7$. Then G has exactly two classes of π-Hall subgroups with representatives

$$\begin{pmatrix} \boxed{\mathrm{GL}_2(2)} & * \\ 0 & \boxed{1} \end{pmatrix} \text{ and } \begin{pmatrix} \boxed{1} & * \\ 0 & \boxed{\mathrm{GL}_2(2)} \end{pmatrix}.$$

The map $\iota : x \in G \mapsto (x^t)^{-1}$ is an automorphism of order 2 of G. It interchanges classes of π-Hall subgroups, hence the group $\widehat{G} = G : \langle \iota \rangle$ does not possess a π-Hall subgroup.

Theorem (Alternating and symmetric groups)

Let π be a set of primes.

1. All possibilities for Sym_n to contain a nonstandard π-Hall subgroup are listed in table below.
2. The following statements are equivalent: $\mathrm{Sym}_n \in C_\pi$; $\mathrm{Sym}_n \in E_\pi$; $\mathrm{Alt}_n \in E_\pi$; $\mathrm{Alt}_n \in C_\pi$.
3. $M \in \mathrm{Hall}_\pi(\mathrm{Alt}_n)$ if and only if there exists $M_0 \in \mathrm{Hall}_\pi(\mathrm{Sym}_n)$ such that $M = M_0 \cap \mathrm{Alt}_n$.

n	$\pi \cap \pi(\mathrm{Sym}_n)$	$H \in \mathrm{Hall}_\pi(Sym_n)$
prime	$\pi((n-1)!)$	Sym_{n-1}
7	$\{2,3\}$	$\mathrm{Sym}_3 \times \mathrm{Sym}_4$
8	$\{2,3\}$	$\mathrm{Sym}_4 \wr \mathrm{Sym}_2$

Theorem (Alternating and symmetric groups)

Let π be a set of primes.

1. All possibilities for Sym_n to contain a nonstandard π-Hall subgroup are listed in table below.
2. The following statements are equivalent: $\mathrm{Sym}_n \in C_\pi$; $\mathrm{Sym}_n \in E_\pi$; $\mathrm{Alt}_n \in E_\pi$; $\mathrm{Alt}_n \in C_\pi$.
3. $M \in \mathrm{Hall}_\pi(\mathrm{Alt}_n)$ if and only if there exists $M_0 \in \mathrm{Hall}_\pi(\mathrm{Sym}_n)$ such that $M = M_0 \cap \mathrm{Alt}_n$.

n	$\pi \cap \pi(\mathrm{Sym}_n)$	$H \in \mathrm{Hall}_\pi(Sym_n)$
prime	$\pi((n-1)!)$	Sym_{n-1}
7	$\{2,3\}$	$\mathrm{Sym}_3 \times \mathrm{Sym}_4$
8	$\{2,3\}$	$\mathrm{Sym}_4 \wr \mathrm{Sym}_2$

Sporadic E_π-groups, $2 \notin \pi$

G	$\pi \cap \pi(G)$	G	$\pi \cap \pi(G)$	G	$\pi \cap \pi(G)$
M_{11}	$\{5, 11\}$	M_{12}	$\{5, 11\}$	M_{22}	$\{5, 11\}$
Ru	$\{7, 29\}$	M_{24}	$\{5, 11\}$ $\{11, 23\}$	M_{23}	$\{5, 11\}$ $\{11, 23\}$
Fi_{23}	$\{11, 23\}$	Fi'_{24}	$\{11, 23\}$	Ly	$\{11, 67\}$
J_1	$\{3, 5\}$ $\{3, 7\}$ $\{3, 19\}$ $\{5, 11\}$	J_4	$\{5, 7\}$ $\{5, 11\}$ $\{5, 31\}$ $\{7, 29\}$ $\{7, 43\}$	$O'N$	$\{3, 5\}$ $\{5, 11\}$ $\{5, 31\}$
Co_1	$\{11, 23\}$	Co_2	$\{11, 23\}$	Co_3	$\{11, 23\}$
B	$\{11, 23\}$ $\{23, 47\}$	M	$\{23, 47\}$ $\{29, 59\}$		

π-Hall subgroups of the sporadic groups, $2 \in \pi$

G	π	Structure of H
M_{11}	$\{2,3\}$	$3^2 : Q_8 \,.\, 2$
	$\{2,3,5\}$	$\mathrm{Alt}_6 \,.\, 2$
M_{22}	$\{2,3,5\}$	$2^4 : \mathrm{Alt}_6$
M_{23}	$\{2,3\}$	$2^4 : (3 \times \mathrm{Alt}_4) : 2$
	$\{2,3,5\}$	$2^4 : \mathrm{Alt}_6$
	$\{2,3,5\}$	$2^4 : (3 \times \mathrm{Alt}_5) : 2$
	$\{2,3,5,7\}$	$\mathrm{L}_3(4) : 2_2$
	$\{2,3,5,7\}$	$2^4 : \mathrm{Alt}_7$
	$\{2,3,5,7,11\}$	M_{22}
M_{24}	$\{2,3,5\}$	$2^6 : 3 \,^{\cdot}\mathrm{Sym}_6$
J_1	$\{2,3\}$	$2 \times \mathrm{Alt}_4$
	$\{2,7\}$	$2^3 : 7$
	$\{2,3,5\}$	$2 \times \mathrm{Alt}_5$
	$\{2,3,7\}$	$2^3 : 7 : 3$
J_4	$\{2,3,5\}$	$2^{11} : (2^6 : 3 \,^{\cdot}\mathrm{Sym}_6)$

Theorem (Revin, 1999)

Let G be a finite group of Lie type over a field of characteristic $p \in \pi$. If H is a π-Hall subgroup of G, then either H is included in a Borel subgroup of G, or H is equal to a parabolic subgroup of G.

Theorem (Gross, 1986, 1993; Revin, 1999)

Let π be a set of primes. Let G be a group of Lie type with the base field $\mathbb{F}_q$ of characteristic $p \in \pi$, denote by B a Borel subgroup of G. Then $|G : B|$ is a π'-number if and only if $\pi \cap \pi(G) \subseteq \pi(q-1) \cup \{p\}$ and $\pi \cap \pi(W) \subseteq \{p\}$, where W is the Weyl group of G. Sets $\pi(W)$ for all finite groups of Lie type are given in table below.

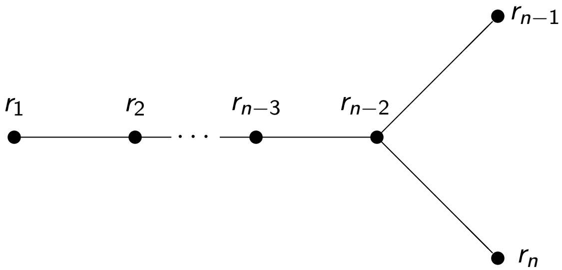
G	$\pi(W)$	G	$\pi(W)$
$A_{n-1}(q)$, $B_n(q)$, $C_n(q)$, $D_n(q)$	$\pi(n!)$	${}^2E_6(q)$, $F_4(q)$, $G_2(q)$, ${}^3D_4(q)$	$\{2,3\}$
${}^2A_{n-1}(q)$	$\pi([n/2]!)$	${}^2D_n(q)$	$\pi((n-1)!)$
$E_7(q)$, $E_8(q)$	$\{2,3,5,7\}$	$E_6(q)$	$\{2,3,5\}$
${}^2B_2(q)$, ${}^2G_2(q)$,	$\{2\}$	${}^2F_4(q)$	$\{2\}$

Parabolic π-Hall subgroups in $D_n^\eta(q)$

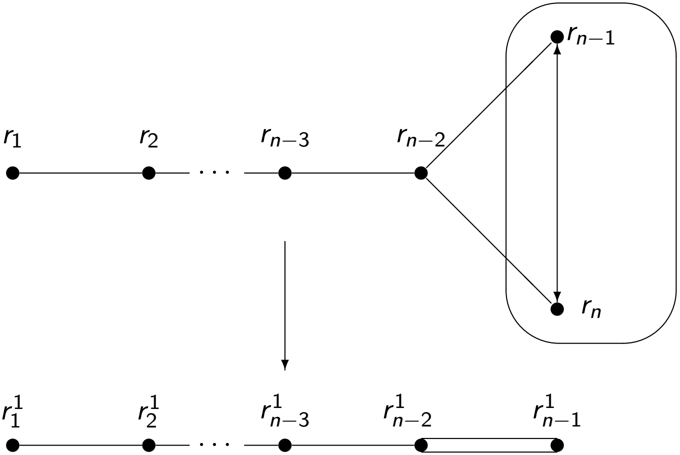
Let π be a set of primes. Let $G = D_n^\eta(q)$ be a group of Lie type with the base field $\mathbb{F}_q$ of characteristic $p \in \pi$. Then G possesses a proper parabolic π-subgroup P with $\pi(|G : P|) \subseteq \pi'$ if and only if one of the following statements holds:

1. $G = D_n^+(q)$, $p = 2$, n is a Fermat prime, $(n, q-1) = 1$, P is conjugate to a parabolic subgroup G_J corresponding to the set $J = \{r_2, r_3, \ldots, r_n\}$ of fundamental roots and $\pi \cap \pi(G) = \pi(P) = \pi(G) \setminus \pi(\frac{q^n-1}{q-1}(q^{n-1}+1))$;
2. $G = D_n^-(q)$, $p = 2$, $n-1$ is a Mersenne prime, $(n-1, q-1) = 1$, P is conjugate to a parabolic subgroup G_J corresponding to the set $J = \{r_2^1, r_3^1, \ldots, r_{n-1}^1\}$ of fundamental roots and $\pi \cap \pi(G) = \pi(P) = \pi(G) \setminus \pi(\frac{q^{n-1}-1}{q-1}(q^n+1))$.

Dynkin diagram for the root system of type D_n.



Dynkin diagram for the root system of type ${}^2D_n$.



Parabolic π-Hall subgroups in $A_{n-1}(q)$

Let π be a set of primes, V an n-dimensional vector space over a field $\mathbb{F}_q$ of characteristic $p \in \pi$, $G = \mathrm{SL}(V) \simeq \mathrm{SL}_n(q)$ be a special linear group. Each parabolic subgroup P of G can be obtained as a stabilizer of

$$0 = V_0 < V_1 < \cdots < V_s = V$$

with $\dim V_i / V_{i-1} = n_i$, $i = 1, \ldots, s$. All cases, when P is a proper π-subgroup, while $\pi(|G : P|) \subseteq \pi'$, are listed in table below. Each parabolic π-Hall subgroup M of $\mathrm{PSL}_n(q) \simeq A_{n-1}(q)$ can be obtained as $P/Z(\mathrm{SL}_n(q))$, where P is a parabolic π-Hall subgroup of $\mathrm{SL}_n(q)$.

n	s	$\{n_1, \dots, n_s\}$	$\pi' \cap \pi(G)$	Other conditions
odd prime	2	$\{1, n-1\}$	$\pi\left(\frac{q^n-1}{q-1}\right)$	$(n, q-1) = 1$
4	2	$\{2\}$	$\pi\left(\frac{(q^3-1)(q^4-1)}{(q-1)(q^2-1)}\right)$	$(6, q-1) = 1$
5	2	$\{2, 3\}$	$\pi\left(\frac{(q^4-1)(q^5-1)}{(q^2-1)(q-1)}\right)$	$(10, q-1) = 1$
5	3	$\{1, 2\}$	$\pi\left(\frac{(q^3-1)(q^4-1)(q^5-1)}{(q-1)(q^2-1)(q-1)}\right)$	$(30, q-1) = 1$
7	2	$\{3, 4\}$	$\pi\left(\frac{(q^3+1)(q^5-1)(q^7-1)}{(q+1)(q-1)(q-1)}\right)$	$(35, q-1) = 1$, $(3, q+1) = 1$
8	2	$\{4\}$	$\pi\left(\frac{(q^3+1)(q^4-1)(q^5-1)(q^7-1)}{(q+1)(q^2-1)(q-1)(q-1)}\right)$	$(70, q-1) = 1$, $(3, q+1) = 1$
11	2	$\{5, 6\}$	$\pi\left(\frac{(q^5+1)(q^7-1)(q^8-1)(q^9-1)(q^{11}-1)}{(q+1)(q-1)(q^4-1)(q^3-1)(q-1)}\right)$	$(462, q-1) = 1$, $(5, q+1) = 1$

Recall that every finite group of Lie type can be obtained in the following way. Consider a simple connected linear algebraic group $\overline{G}$ defined over the algebraic closure $\overline{\mathbb{F}}_p$ of a finite field of order p. Suppose that σ is a surjective endomorphism of $\overline{G}$ such that the set of σ-stable points $\overline{G}_\sigma$ is finite. Then $G = O^{p'}(\overline{G}_\sigma)$ is a finite group of Lie type and every group of Lie type can be obtained in this way. Let $\overline{T}$ be a maximal σ-stable torus of $\overline{G}$, then $T = \overline{T} \cap G = \overline{T}_\sigma \cap G$ is called a maximal torus of G and $N(G,T) = N_{\overline{G}}(\overline{T}) \cap G$ is called the algebraic normalizer of T. Notice that $N(G,T) \leq N_G(T)$, but the inclusion may be proper. The structure of T and $N(G,T)$ is determined in by the conjugacy classes of the Weyl group W of $\overline{G}$.

Namely, $|G|$ can be written as $\frac{1}{d}F(q)$, where d is a divisor of the order of Schur multiplier of G. If G is a split group and a maximal torus T corresponds to an element $w \in W$, then

$$|T| = \frac{1}{d}f_w(q), \text{ and } N(G,T)/T \simeq C_W(w),$$

where $f_w(q)$ is the characteristic polynomial of w (recall that W is a finite group of orthoronal linear transformation of a Eucledean space generated by the roots). In particular, all modules of eigenvalues of w are equal to 1, hence $|T| \leqslant \frac{1}{d}(q+1)^n$, where n is the rank of $\overline{G}$. If G is a twisted group, then the description is more complecated, but anyway is known.

π-Hall subgroups with 2 or 3 not in π (Gross 1993, 1995; Revin, Vdovin 2002, 2006)

Let π be a set of primes such that either 2, or 3 is not in π. Assume that G is a finite group of Lie type defined over a field of characteristic $p \notin \pi$. Suppose G possesses a π-Hall subgroup H. Then one of the following statements holds:

1. There exist a maximal torus T such that $H \leq N(G, T)$.
2. $G = {}^2G_2(3^{2n+1})$, $\pi \cap \pi(G) = \{2, 7\}$, $|G|_{\{2,7\}} = 56$ and H is a Frobenius group of order 56 (the normalizer of a Sylow 2-subgroup).

π-Hall subgroups with 2 or 3 not in π (Gross 1993, 1995; Revin, Vdovin 2002, 2006)

Let π be a set of primes such that either 2, or 3 is not in π. Assume that G is a finite group of Lie type defined over a field of characteristic $p \notin \pi$. Suppose G possesses a π-Hall subgroup H. Then one of the following statements holds:

1. There exist a maximal torus T such that $H \leq N(G, T)$.
2. $G = {}^2G_2(3^{2n+1})$, $\pi \cap \pi(G) = \{2, 7\}$, $|G|_{\{2,7\}} = 56$ and H is a Frobenius group of order 56 (the normalizer of a Sylow 2-subgroup).

π-Hall subgroups with 2 or 3 not in π (Gross 1993, 1995; Revin, Vdovin 2002, 2006)

Let π be a set of primes such that either 2, or 3 is not in π. Assume that G is a finite group of Lie type defined over a field of characteristic $p \notin \pi$. Suppose G possesses a π-Hall subgroup H. Then one of the following statements holds:

1. There exist a maximal torus T such that $H \leq N(G, T)$.
2. $G = {}^2G_2(3^{2n+1})$, $\pi \cap \pi(G) = \{2, 7\}$, $|G|_{\{2,7\}} = 56$ and H is a Frobenius group of order 56 (the normalizer of a Sylow 2-subgroup).

Lemma

Assume that $G \simeq \mathrm{SL}_2(q) \simeq \mathrm{SL}_2^{\eta}(q) \simeq \mathrm{Sp}_2(q)$, where q is a power of an odd prime p, and $\varepsilon = (-1)^{(q-1)/2}$. Then $\mathrm{P}G \in E_\pi$ if and only if one of the cases from the table below occurs. Moreover, if G satisfies E_π, then G contains one, two or three classes of conjugate π-Hall subgroups, i. e., $k_\pi(G) = k_\pi(\mathrm{P}G) \in \{1, 2, 3\}$.

$\pi \cap \pi(G)$	H	classes	Conditions
$\subseteq \pi(q-\varepsilon)$	D_m	1	$m = (q-\varepsilon)_\pi$
$\{2,3\}$	Alt_4	1	$(q^2-1)_{\{2,3\}} = 24$
$\{2,3\}$	Sym_4	2	$(q^2-1)_{\{2,3\}} = 48$
$\{2,3,5\}$	Alt_5	2	$(q^2-1)_{\{2,3,5\}} = 120$

Theorem (Revin, Vdovin, to appear)

Let π be a set of primes such that $2, 3 \in \pi$. Suppose V is a linear, unitary, or symplectic space of dimension n with the base field $\mathbb{F}_q$ of characteristic $p \notin \pi$. Assume that G is chosen so that $\Omega(V) \leq G \leq I(V)$, and G possesses a π-Hall subgroup H. Then one of the following statements holds.

1. H stabilizes a decomposition $V = V_1 \perp \ldots \perp V_k$ into a direct sum of pairwise orthogonal nondegenerate (arbitrary if V is linear) subspaces V_i, and $\dim(V_i) \leqslant 2$ for $i = 1, \ldots, k$.
2. V is a linear or a unitary space, $\dim(V) = 4$, $I(V) = \mathrm{GL}^\eta(V)$, $|\mathrm{P}G : \mathrm{PSL}^\eta(V)| \leqslant 2$, $\pi \cap \pi(G) = \{2, 3, 5\}$, $q \equiv 5\eta \pmod 8$ (in particular $|\mathrm{PGL}^\eta(V) : \mathrm{PSL}^\eta(V)| = 4$ and $\mathrm{P}G \neq \mathrm{PGL}^\eta(V)$), $(q+\eta)_3 = 3$, $(q^2+1)_5 = 5$. Moreover $H \simeq 4\,.\,2^4\,.\,\mathrm{Sym}_6$, if $|\mathrm{P}G : \mathrm{PSL}^\eta(V)| = 2$, and $H \simeq 4\,.\,2^4\,.\,\mathrm{Alt}_6$, if $\mathrm{P}G = \mathrm{PSL}^\eta(V)$.

Theorem (Revin, Vdovin, to appear)

Let π be a set of primes such that $2, 3 \in \pi$. Suppose V is a linear, unitary, or symplectic space of dimension n with the base field $\mathbb{F}_q$ of characteristic $p \notin \pi$. Assume that G is chosen so that $\Omega(V) \leq G \leq I(V)$, and G possesses a π-Hall subgroup H. Then one of the following statements holds.

1. H stabilizes a decomposition $V = V_1 \perp \ldots \perp V_k$ into a direct sum of pairwise orthogonal nondegenerate (arbitrary if V is linear) subspaces V_i, and $\dim(V_i) \leqslant 2$ for $i = 1, \ldots, k$.
2. V is a linear or a unitary space, $\dim(V) = 4$, $I(V) = \mathrm{GL}^\eta(V)$, $|\mathrm{P}G : \mathrm{PSL}^\eta(V)| \leqslant 2$, $\pi \cap \pi(G) = \{2, 3, 5\}$, $q \equiv 5\eta \pmod 8$ (in particular $|\mathrm{PGL}^\eta(V) : \mathrm{PSL}^\eta(V)| = 4$ and $\mathrm{P}G \neq \mathrm{PGL}^\eta(V)$), $(q+\eta)_3 = 3$, $(q^2+1)_5 = 5$. Moreover $H \simeq 4 \,.\, 2^4 \,.\, \mathrm{Sym}_6$, if $|\mathrm{P}G : \mathrm{PSL}^\eta(V)| = 2$, and $H \simeq 4 \,.\, 2^4 \,.\, \mathrm{Alt}_6$, if $\mathrm{P}G = \mathrm{PSL}^\eta(V)$.

Theorem (Revin, Vdovin, to appear)

Let π be a set of primes such that $2, 3 \in \pi$. Suppose V is a linear, unitary, or symplectic space of dimension n with the base field $\mathbb{F}_q$ of characteristic $p \notin \pi$. Assume that G is chosen so that $\Omega(V) \leq G \leq I(V)$, and G possesses a π-Hall subgroup H. Then one of the following statements holds.

1. H stabilizes a decomposition $V = V_1 \perp \ldots \perp V_k$ into a direct sum of pairwise orthogonal nondegenerate (arbitrary if V is linear) subspaces V_i, and $\dim(V_i) \leqslant 2$ for $i = 1, \ldots, k$.
2. V is a linear or a unitary space, $\dim(V) = 4$, $I(V) = \mathrm{GL}^\eta(V)$, $|\mathrm{P}G : \mathrm{PSL}^\eta(V)| \leqslant 2$, $\pi \cap \pi(G) = \{2, 3, 5\}$, $q \equiv 5\eta \pmod 8$ (in particular $|\mathrm{PGL}^\eta(V) : \mathrm{PSL}^\eta(V)| = 4$ and $\mathrm{P}G \neq \mathrm{PGL}^\eta(V)$), $(q+\eta)_3 = 3$, $(q^2+1)_5 = 5$. Moreover $H \simeq 4 \,.\, 2^4 \,.\, \mathrm{Sym}_6$, if $|\mathrm{P}G : \mathrm{PSL}^\eta(V)| = 2$, and $H \simeq 4 \,.\, 2^4 \,.\, \mathrm{Alt}_6$, if $\mathrm{P}G = \mathrm{PSL}^\eta(V)$.

Theorem (Revin, Vdovin, to appear)

Let π be a set of primes such that $2, 3 \in \pi$, V be a vector space with a nondegenerate symmetric bilinear form over a field $\mathbb{F}_q$ of odd characteristic $p \notin \pi$, G ischose so that $\Omega(V) \leq G \leq I(V)$. Assume also that G possesses a π-Hall subgroup H. Then one of the following statements holds:

1. There exists a H-invariant decomposition
$$V = V_1 \oplus \ldots \oplus V_m$$
of V into an orthogonal sum of nondegenerate subspaces such that $\dim(V_i) \leqslant 4$ for $i = 1, \ldots, m$.
2. $\dim(V) = 7$, $G = \Omega(V)$, $H \simeq \Omega_7(2)$.
3. $\dim(V) = 8$, $\eta(V) = +$, $G = \Omega(V)$, $H \simeq 2 . \Omega_8^+(2)$.
4. $\dim(V) = 9$, $G = \Omega(V)$, $H \simeq 2 . \Omega_8^+(2) . 2$.

Theorem (Revin, Vdovin, to appear)

Let π be a set of primes such that $2, 3 \in \pi$, V be a vector space with a nondegenerate symmetric bilinear form over a field $\mathbb{F}_q$ of odd characteristic $p \notin \pi$, G ischose so that $\Omega(V) \leq G \leq I(V)$. Assume also that G possesses a π-Hall subgroup H. Then one of the following statements holds:

1. There exists a H-invariant decomposition
$$V = V_1 \oplus \ldots \oplus V_m$$
of V into an orthogonal sum of nondegenerate subspaces such that $\dim(V_i) \leqslant 4$ for $i = 1, \ldots, m$.
2. $\dim(V) = 7$, $G = \Omega(V)$, $H \simeq \Omega_7(2)$.
3. $\dim(V) = 8$, $\eta(V) = +$, $G = \Omega(V)$, $H \simeq 2 . \Omega_8^+(2)$.
4. $\dim(V) = 9$, $G = \Omega(V)$, $H \simeq 2 . \Omega_8^+(2) . 2$.

Theorem (Revin, Vdovin, to appear)

Let π be a set of primes such that $2, 3 \in \pi$, V be a vector space with a nondegenerate symmetric bilinear form over a field $\mathbb{F}_q$ of odd characteristic $p \notin \pi$, G ischose so that $\Omega(V) \leq G \leq I(V)$. Assume also that G possesses a π-Hall subgroup H. Then one of the following statements holds:

1. There exists a H-invariant decomposition
$$V = V_1 \oplus \ldots \oplus V_m$$
of V into an orthogonal sum of nondegenerate subspaces such that $\dim(V_i) \leqslant 4$ for $i = 1, \ldots, m$.
2. $\dim(V) = 7$, $G = \Omega(V)$, $H \simeq \Omega_7(2)$.
3. $\dim(V) = 8$, $\eta(V) = +$, $G = \Omega(V)$, $H \simeq 2 \,.\, \Omega_8^+(2)$.
4. $\dim(V) = 9$, $G = \Omega(V)$, $H \simeq 2 \,.\, \Omega_8^+(2) \,.\, 2$.

Corollary

Let π be a set of primes with $2, 3 \in \pi$. Assume that G is a classical group over a field of characteristic $p \notin \pi$ and the dimention of the natural G-module is at least 18. Suppose also that G possesses a π-Hall subgroup H. Then there exists a maximal torus T of G such that $H \leq N(G, T)$.

Lemma (Revin, Vdovin, to appear)

Let π be a set of primes with $2, 3 \in \pi$. Assume that G is an exceptional group over a field of characteristic $p \notin \pi$. Suppose also that G possesses a π-Hall subgroup H. Then one of the following statements holds:

1. There exists a maximal torus T of G such that $H \leq N(G, T)$.
2. $G \simeq G_2(q)$, $\pi \cap \pi(G) = \{2, 3, 7\}$, $(q^2 - 1)_{\{2,3,7\}} = 24$, $(q^4 + q^2 + 1)_7 = 7$, $H \simeq G_2(2)$.

Lemma (Revin, Vdovin, to appear)

Let π be a set of primes with $2, 3 \in \pi$. Assume that G is an exceptional group over a field of characteristic $p \notin \pi$. Suppose also that G possesses a π-Hall subgroup H. Then one of the following statements holds:

1. There exists a maximal torus T of G such that $H \leq N(G, T)$.
2. $G \simeq G_2(q)$, $\pi \cap \pi(G) = \{2, 3, 7\}$, $(q^2 - 1)_{\{2,3,7\}} = 24$, $(q^4 + q^2 + 1)_7 = 7$, $H \simeq G_2(2)$.

Lemma (Revin, Vdovin, to appear)

Let π be a set of primes with $2, 3 \in \pi$. Assume that G is an exceptional group over a field of characteristic $p \notin \pi$. Suppose also that G possesses a π-Hall subgroup H. Then one of the following statements holds:

1. There exists a maximal torus T of G such that $H \leq N(G, T)$.
2. $G \simeq G_2(q)$,$\pi \cap \pi(G) = \{2, 3, 7\}$, $(q^2 - 1)_{\{2,3,7\}} = 24$, $(q^4 + q^2 + 1)_7 = 7$, $H \simeq G_2(2)$.

Theorem (Revin, Vdovin, to appear)

Let S be a finite simple group, π be a set of primes. Then one of the following statements holds:

1. If $2 \notin \pi$ then $k_\pi(S) \in \{0, 1\}$.
2. If $3 \notin \pi$ then $k_\pi(S) \in \{0, 1, 2\}$.
3. If $2, 3 \in \pi$ then $k_\pi(S) \in \{0, 1, 2, 3, 4, 9\}$.

In particular, if $S \in E_\pi$, then $k_\pi(S)$ is a π-number.

Assume that $X \in E_\pi$ is such that $k = k_\pi(X) > 1$. Suppose $p \in \pi'$. Denote a cyclic subgroup of order p of Sym_p by Y. Consider $G = X \wr Y$ and let

$$M \simeq \underbrace{X \times \cdots \times X}_{p \text{ times}}$$

be the base of the wreath product. It is clear that $k_\pi(M) = k^p$. Since M is a normal subgroup of G and $|G : M| = p$ is a π'-number, then $\mathrm{Hall}_\pi(G) = \mathrm{Hall}_\pi(M)$. The subgroup Y acts on the set of classes of conjugate π-Hall subgroups of M. Applying Burnside formula to this action it is easy to show that

$$k_\pi(G) = \frac{k^p + (p-1)k}{p}.$$

Now assume that $\pi = \{2,3\}$ and $X = \mathrm{SL}_3(2)$. Then $k_\pi(X) = 2$. Since $p \in \pi'$ can be taken arbitrary large and $(2^p - 2)/p + 2$ tends to infinity when p tends to infinity, we obtain that for nonsimple group G the number $k_\pi(G)$ is not bounded in general. Moreover, if we take $p = 7$, then $k_\pi(G) = 20 \notin \pi$.

THANK YOU FOR ATTENTION