A solvable group isospectral to $S_4(3)$

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G is a finite group.

The set of element orders

$$\omega(G) = \{ n \in \mathbb{N} \mid \exists\ g \in G : o(g) = n\},$$

called the spectrum of G, is uniquely determined by the set $\mu(G)$ of its maximal (under divisibility) elements.

The spectrum carries much information about the group.

Example: $\mu(G) = \{2, 3, 5\} \quad \Rightarrow \quad G \cong \mathrm{Alt}_5$

What other finite simple groups have this property?

Groups G and H are isospectral if $\omega(G) = \omega(H)$.

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Theorem [M. S. Lucido, A. R. Moghaddamfar; 2004] If G is a nonabelian simple group isospectral to a solvable group then $G \cong L_3(3), U_3(3), S_4(3), Alt_{10}$.

Theorem [A. M. Staroletov; 2008] There exists no solvable group isospectral to Alt_{10}.

The existence of solvable groups isospectral to $L_3(3)$ and $U_3(3)$ was proven by V. D. Mazurov and M. R. Zinovieva.

The prime graph $\Gamma(G)$ of G:

Vertices: $\pi(G)$ (the prime divisors of $|G|$).

Edges: $p - q \iff pq \in \omega(G)$

- The knowledge of $\omega(G)$ allows one to construct $\Gamma(G)$.
- Isospectral groups have equal prime graphs.

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- The knowledge of $\omega(G)$ allows one to construct $\Gamma(G)$.
- Isospectral groups have equal prime graphs.

$s(G)$ is the number of connected components of $\Gamma(G)$.

- $s(G) \leqslant 6$ for all finite groups G.
- $s(G) = 6 \quad \Rightarrow \quad G \cong J_4$.

Groups G with $s(G) > 1$ have a very restricted structure.

Theorem [K. W. Gruenberg, O. Kegel; 1975] If G is a finite solvable group with $s(G) > 1$ then $s(G) = 2$ and one of the following conditions holds:

• $G = AB$ is a Frobenius group and the two components of $\Gamma(G)$ are the complete graphs on $\pi(A)$ and $\pi(B)$;

• $G = PAB$ is a doubly Frobenius group (i.e., P and PA and normal in G; PA and AB are Frobenius groups) and the two components of $\Gamma(G)$ are the complete graphs on $\pi(PB)$ and $\pi(A)$.

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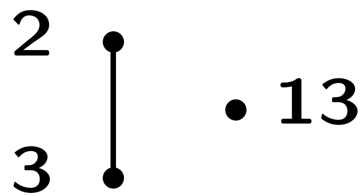
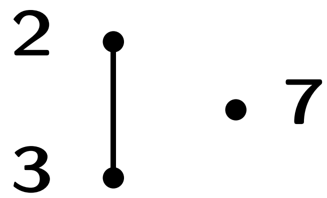
The prime graphs of $\mathrm{L}_3(3)$ and $\mathrm{U}_3(3)$ are disconnected:

$\mu(\mathrm{L}_3(3)) = \{6, 8, 13\}$, $\Gamma(\mathrm{L}_3(3))$:



$\mu(\mathrm{U}_3(3)) = \{7, 8, 12\}$, $\Gamma(\mathrm{U}_3(3))$:



In either case, there exists an isospectral Frobenius group:

$\mu(B) = \{6, 8\}$, where $B = 2.\mathrm{Sym}_4 \leqslant \mathrm{SL}_2(13^2)$ acts fixed-point freely on A of order 13^4. Hence, $\mu(AB) = \{6, 8, 13\}$.

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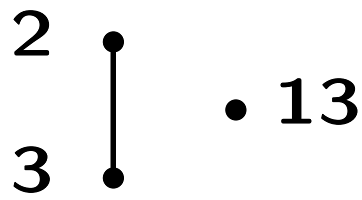

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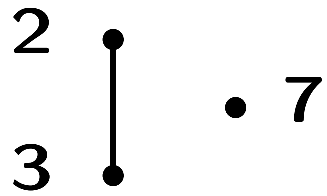

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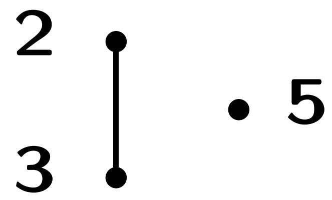
The case of $S_4(3)$ is more difficult.

$\mu(S_4(3)) = \{5, 9, 12\}$, $\quad\quad \Gamma(S_4(3))$:



There exists no Frobenius group AB isospectral to $S_4(3)$.

Indeed,

- if $\pi(A) = \{2, 3\}$ then $\quad 4, 9 \in \omega(A) \Rightarrow 36 \in \omega(A)$, since A is nilpotent;

- if $\pi(B) = \{2, 3\}$ then $\quad 2, 9 \in \omega(B) \Rightarrow 18 \in \omega(B)$, since B has a central involution.

Is there a doubly Frobenius group isospectral to $S_4(3)$?

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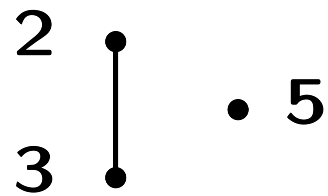

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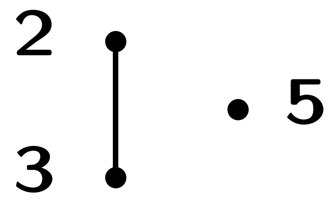

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If G is a doubly Frobenius group with $\mu(G) = \{5, 9, 12\}$ then $G = PAB$, where

- P should be a 3-group of exponent 9;
- AB should be Frobenius of the form 5 : 4;
- A must act on P fixed-point freely;
- $C_P(b^2)$ must have exponent 3, where $B = \langle b \rangle$.

Construction:

The group $AB = \langle a, b \mid a^5 = b^4 = 1, a^b = a^2 \rangle$ has a unique faithful irreducible canonical representation in characteristic 3.

$$a \to \alpha = \begin{pmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ -1 & -1 & -1 & -1 \end{pmatrix}, \quad b \to \beta = \begin{pmatrix} 1 & . & . & . \\ . & . & 1 & . \\ -1 & -1 & -1 & -1 \\ . & 1 & . & . \end{pmatrix}.$$

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Three 16×16-matrices over $\mathbb{F}_3$ in the block form:

$$a \to \mathrm{diag}(\alpha, \alpha, \alpha, \alpha), \qquad b \to \mathrm{diag}(\beta, \beta, \beta, \beta),$$

$$c = \begin{pmatrix} 1 & \beta & \beta^2 & . \\ . & 1 & . & -\beta^2 \\ . & . & 1 & \beta^3 \\ . & . & . & 1 \end{pmatrix}$$

The group $H = \langle a, b, c \rangle$ acts on the row space $V = (\mathbb{F}_3)^{16}$.

The semidirect product $G = V \rtimes H$ is the required doubly Frobenius group PAB, where $A = \langle a \rangle$, $B = \langle b \rangle$, and $P = V \rtimes \langle c \rangle^A$ is of nilpotency class 3.

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The group G constructed above can be represented by 17×17-matrices over $\mathbb{F}_3$. It has order

$$5\,648\,590\,729\,620 = 2^2 \cdot 3^{24} \cdot 5,$$

and is believed to be the smallest solvable group isospectral to $\mathrm{S}_4(3)$.

Remark The proof that G is isospectral to $\mathrm{S}_4(3)$ does not depend on computer calculations. However, the first evidence for the existence of this group was obtained with the aid of the computer algebra system GAP.

Theorem A finite simple nonabelian group G is isospectral to a solvable group if and only if $G \cong \mathrm{L}_3(3), \mathrm{U}_3(3), \mathrm{S}_4(3)$.

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Sketch of the proof:

17-dimensional representation of G in block form:

$$a \to \mathrm{diag}(1, \alpha, \alpha, \alpha, \alpha), \qquad b \to \mathrm{diag}(1, \beta, \beta, \beta, \beta),$$

$$c \to \left(\begin{array}{c|cccc} 1 & . & . & . & . \\ \hline . & 1 & \beta & \beta^2 & . \\ . & . & 1 & . & -\beta^2 \\ . & . & . & 1 & \beta^3 \\ . & . & . & . & 1 \end{array}\right), \qquad d = \left(\begin{array}{c|cccc} 1 & \delta & . & . & . \\ \hline . & 1 & . & . & . \\ . & . & 1 & . & . \\ . & . & . & 1 & . \\ . & . & . & . & 1 \end{array}\right),$$

where $\delta = (1, 0, 0, 0) \in (\mathbb{F}_3)^4$.

We have $G = \langle a, b, c, d \rangle$.

Step 1: P has exponent 9.

(Recall that $G = PAB$, and $P = \mathrm{O}_3(G)$.)

Each element of P has the form

$$\left(\begin{array}{c|cccc} 1 & \delta_1 & \delta_2 & \delta_3 & \delta_4 \\ \hline . & 1 & \mu_1 & \mu_3 & \nu \\ . & . & 1 & . & \mu_4 \\ . & . & . & 1 & \mu_2 \\ . & . & . & . & 1 \end{array}\right).$$

P is naturally embedded into into the upper unitriangular 5×5-matrix group over $\mathrm{M}_4(\mathbb{F}_3)$.

Lemma Let

$$X = \begin{pmatrix} 1 & \chi_1 & \eta_1 & \theta_1 & \tau_1 \\ . & 1 & \chi_2 & \eta_2 & \theta_2 \\ . & . & 1 & \chi_3 & \eta_3 \\ . & . & . & 1 & \chi_4 \\ . & . & . & . & 1 \end{pmatrix}$$

be an upper unitriangular 5×5-matrix over a (not necessarily commutative) ring of characteristic 3. Then $X^9 = 1$ and $|X| < 9$ iff

$$\begin{aligned} \chi_1\chi_2\chi_3 &= 0, \\ \chi_2\chi_3\chi_4 &= 0, \\ \chi_1\chi_2\eta_3 + \chi_1\eta_2\chi_4 + \eta_1\chi_3\chi_4 &= 0. \end{aligned}$$

P has an element such that $\delta_1(\mu_1\mu_4 + \mu_3\mu_2) \neq 0$.

Step 2: A acts fixed-point freely on P.

We have $P = V \rightthreetimes \langle c \rangle^A$.

$V = (\mathbb{F}_3)^{16} = W \oplus W \oplus W \oplus W$,

where W is the canonical module for $AB = 5:4$.

Due to the special choice of the matrix

$$c = \begin{pmatrix} 1 & \beta & \beta^2 & . \\ . & 1 & . & -\beta^2 \\ . & . & 1 & \beta^3 \\ . & . & . & 1 \end{pmatrix},$$

one shows that $\langle c \rangle^A = 3^4.3^4$ is of nilpotency class 2 with central factors isomorphic to W as $\mathbb{F}_3 AB$-modules.

Step 3: $C_P(b^2)$ has exponent 3.

Every element of $C_P(b^2)$ has the form

$$\left(\begin{array}{c|cccc} 1 & \delta_1 & \delta_2 & \delta_3 & \delta_4 \\ \hline . & 1 & \mu_1 & \mu_3 & \nu \\ . & . & 1 & . & \mu_4 \\ . & . & . & 1 & \mu_2 \\ . & . & . & . & 1 \end{array}\right)$$

with $\delta_i \in C_W(b^2)$ and $\mu_i, \nu \in C_M(b^2)$, where

W is the canonical module for $AB = 5:4$, and

$M = \mathrm{M}_4(\mathbb{F}_3)$ viewed as a module for AB with action by conjugation.

It can be shown that $C_P(b^2)$ has exponent 3 iff

$$w\,\psi(u) = 0 \qquad (*)$$

for all $w, u \in C_W(b^2)$, where

$$\psi : C_W(b^2) \to M$$

is a certain quadratic map of $\mathbb{F}_3 B$-modules.

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the identity $(*)$ can be shown to hold.

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