Logically and algebraically homogeneous groups

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Joint work with B.I. Plotkin[1], E. Plotkin[2]

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We study algebraic structures within the frames of Logical Geometry.

What is Logical Geometry?

Classical Algebraic Geometry

$$f(x_1, \ldots, x_n) = 0,$$

$f(x_1, \ldots, x_n)$ are elements of the algebra $K[x_1, \ldots, x_n]$.

$K[x_1, \ldots, x_n]$ is the free algebra in the variety $Com - K$.

Equational Universal Algebraic Geometry

$$w(x_1, \ldots, x_n) = w'(x_1, \ldots, x_n),$$

w, w' are elements of the free algebra $W(x_1, \ldots, x_n)$ in the fixed variety of algebras.

Research in this area started with a series of papers by B. Plotkin, G. Baumslag, O. Kharlampovich, A. Myasnikov, and V. Remeslennikov.

algebraic geometry over

- groups (free groups, solvable groups,),
- associative algebras,
- Lie algebras,
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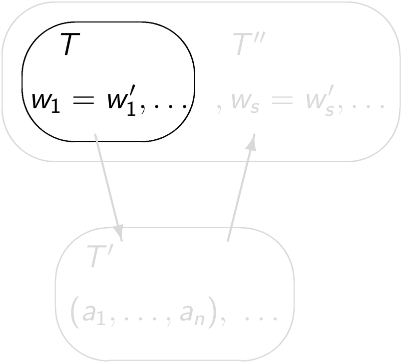
Galois correspondence

Algebra $W(x_1, \ldots, x_n)$



Affine space A^n

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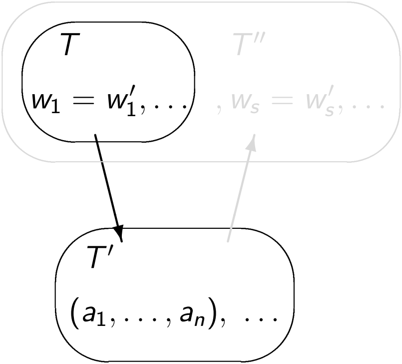

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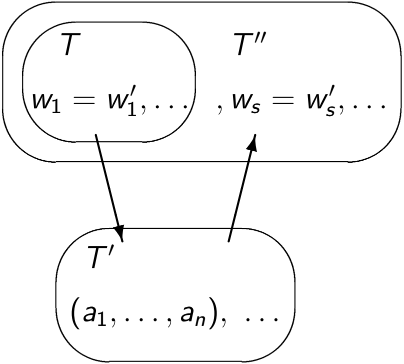

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Logical Geometry

Logical Geometry studies solutions of systems of first order logic formulas "over" equalities $w = w'$, $w, w' \in W(x_1, \ldots, x_n)$.

$$\left(\exists x_1(x_1 = x_2^3)\right) \vee \left(x_1x_2 = x_3\right)$$

We do not use the other predicates (or relations):

$$\left(\exists x_1(x_1 > x_2)\right) \vee \left(x_1x_2 = x_3\right).$$

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Logical Geometry

- Algebra
- Geometry
- Logic and Model Theory

Homogeneity

$$\swarrow \qquad \searrow$$

Algebraic $<$ *Logical*

$$\begin{tikzcd} & \textbf{Homogeneity} \arrow[dl] \arrow[dr] & \\ \textit{Algebraic} & < & \textit{Logical} \end{tikzcd}$$

Homogeneity

$$\swarrow \qquad \searrow$$

Algebraic $<$ *Logical*

Algebraically homogeneous groups

Definition

A group G is called algebraically homogeneous if every isomorphism between two of its finitely generated subgroups can be extended up to an automorphism of G.

Let

- $F(X)$ be a free group, $X = \{x_1, \ldots, x_n\}$,
- G be a group,
- $\mu_1, \mu_2 : F(X) \to G$ be homomorphisms (points in the affine space G^n: $\mu = (g_1, \ldots, g_n)$, $\mu(x_i) = g_i$).

Definition

A group G is algebraically homogeneous if for every two homomorphisms $\mu_1, \mu_2 : F(X) \to G$ whenever $Ker \mu_1 = Ker \mu_2$ then there is an automorphism σ if G such that $\sigma(\mu_1) = \mu_2$.

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Logically homogeneous groups

MT-Types

Let

- G be a group,
- $\mu = (g_1, \dots g_n)$ be a point in G^n.

Definition

A complete model-theoretical type of a point $\mu = (g_1, \dots, g_n)$ is the set of all first order logic formulas (in a language $\mathbb{L}$) in free variables $x_1, \dots, x_n$ which hold true on the point μ:

$$tp^G(\mu) = \{u(x_1, \dots, x_n, y_1, \dots) \in \mathbb{L} \mid G \models u(g_1, \dots, g_n)\}.$$

$$\mathbb{L} = \{\wedge, \vee, \neg, \exists, =, \cdot, \dots\}.$$

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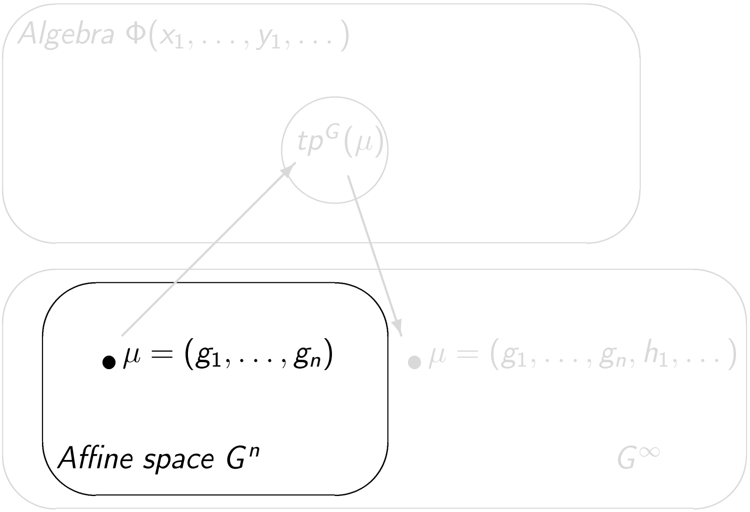
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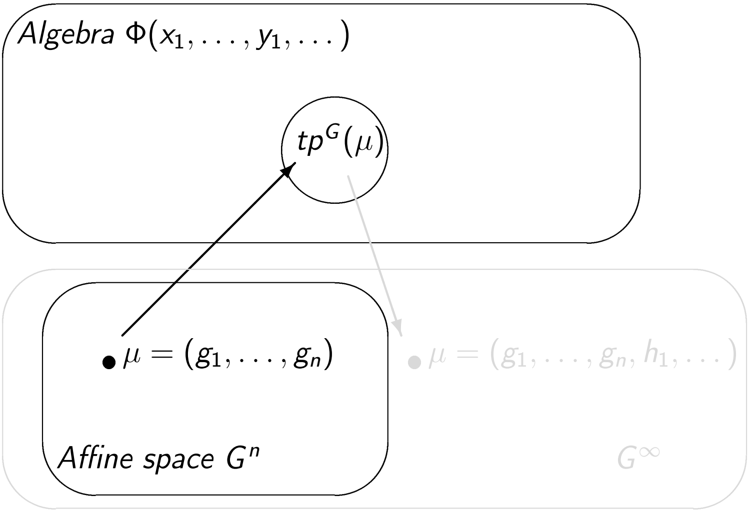
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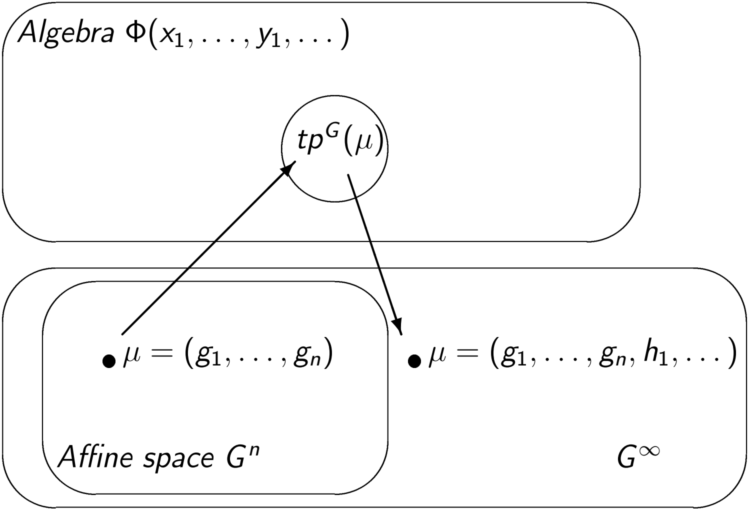
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LG-Types

Definition (B. Plotkin)

A logically-geometrical type of a point $\mu = (g_1, \ldots, g_n)$ is the set of all first order logic formulas in variables $x_1, \ldots, x_n$ (not necessarily free) which hold true on the point μ.

$$LKer(\mu) = \{u(x_1, \ldots, x_n) \in \mathbb{L}^* \mid G \models u(g_1, \ldots, g_n)\}.$$

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Logically homogeneous groups. Some results

Theorem (C. Perin-R. Sklinos, A. Ould Houcine)

Finitely generated free non-abelian group is logically homogeneous.

Theorem (G. Zhitomirskii)

Finitely generated free nilpotent groups are logically homogeneous.

Problem

Is a finitely generated free solvable group logically homogeneous?

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Weakly homogeneous groups

Definition (G. Zhitomirskii)

A group G is called weakly (algebraically) homogeneous if for every isomorphism $\varphi : H_1 \longrightarrow H_2$ between two of its finitely generated subgroups H_1 and H_2, the following condition is satisfied: if φ itself and its inverse map $\varphi^{-1} : H_2 \longrightarrow H_1$ both can be extended to endomorphisms of G then φ can be extended to an automorphism of G.

Theorem (G. Zhitomirskii)

Every weakly homogeneous finitely generated free group is logically homogeneous.

Proposition

Free solvable group of rank 2 is not weakly homogeneous.

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