New progress on factorized groups and subgroup permutability

Paz Arroyo-Jordá

Instituto Universitario de Matemática Pura y Aplicada
Universidad Politécnica de Valencia, Spain

Groups St Andrews 2013 in St Andrews
St Andrews, 3rd-11th August 2013

in collaboration with M. Arroyo-Jordá, A. Martínez-Pastor and M.D. Pérez-Ramos

Factorized groups

$\boxed{\text{All groups considered will be finite}}$

Factorized groups: A and B subgroups of a group G

$$G = AB$$

- How the structure of the factors A and B affects the structure of the whole group G?
- How the structure of the group G affects the structure of A and B?

Factorized groups

Natural approach: Classes of groups

A class of groups is a collection $\mathcal{F}$ of groups with the property that if $G \in \mathcal{F}$ and $G \cong H$, then $H \in \mathcal{F}$

Factorized groups

Natural approach: Classes of groups

A class of groups is a collection $\mathcal{F}$ of groups with the property that if $G \in \mathcal{F}$ and $G \cong H$, then $H \in \mathcal{F}$

Question

Let $\mathcal{F}$ be a class of groups and $G = AB$ a factorized group:

- $A, B \in \mathcal{F} \implies G \in \mathcal{F}$?
- $G \in \mathcal{F} \implies A, B \in \mathcal{F}$?

Definitions

- A formation is a class $\mathcal{F}$ of groups with the following properties:
  - Every homomorphic image of an $\mathcal{F}$-group is an $\mathcal{F}$-group.
  - If G/M and $G/N \in \mathcal{F}$, then $G/(M \cap N) \in \mathcal{F}$
- $\mathcal{F}$ a formation: the $\mathcal{F}$-residual $G^{\mathcal{F}}$ of G is the smallest normal subgroup of G such that $G/G^{\mathcal{F}} \in \mathcal{F}$
- The formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$, then $G \in \mathcal{F}$.

Starting point

$$\boxed{G = AB: \quad A, B \in \mathcal{U}, \ \ A, B \trianglelefteq G \ \not\Longrightarrow \ G \in \mathcal{U}}$$

Example

$Q = \langle x, y \rangle \cong Q_8, \quad V = \langle a, b \rangle \cong C_5 \times C_5$
$G = [V]Q$ the semidirect product of V with Q
$G = AB$ with $A = V\langle x \rangle$ and $B = V\langle y \rangle$
$A, B \in \mathcal{U}, \ \ A, B \trianglelefteq G, \ \ G \notin \mathcal{U}$

Starting point

$$\boxed{G = AB : \quad A, B \in \mathcal{U}, \ \ A, B \trianglelefteq G \ \not\Longrightarrow \ G \in \mathcal{U}}$$

Example

$Q = \langle x, y \rangle \cong Q_8, \quad V = \langle a, b \rangle \cong C_5 \times C_5$
$G = [V]Q$ the semidirect product of V with Q
$G = AB$ with $A = V\langle x \rangle$ and $B = V\langle y \rangle$
$A, B \in \mathcal{U}, \ \ A, B \trianglelefteq G, \ \ G \notin \mathcal{U}$

$$G = AB : \quad A, B \in \mathcal{U}, \ \ A, B \trianglelefteq G + \boxed{\text{additional conditions}} \ \Longrightarrow G \in \mathcal{U}$$

- (Baer, 57) $G' \in \mathcal{N}$
- (Friesen,71) $(|G : A|, |G : B|) = 1$

Permutability properties

If $G = AB$ is a central product of the subgroups A and B, then:

$$A, B \in \mathcal{U} \implies G \in \mathcal{U}$$

More generally, if $\mathcal{F}$ is any formation:

$$A, B \in \mathcal{F} \implies G \in \mathcal{F}$$

(In particular, this holds when $G = A \times B$ is a direct product.)

Permutability properties

If $G = AB$ is a central product of the subgroups A and B, then:

$$A, B \in \mathcal{U} \implies G \in \mathcal{U}$$

More generally, if $\mathcal{F}$ is any formation:

$$A, B \in \mathcal{F} \implies G \in \mathcal{F}$$

(In particular, this holds when $G = A \times B$ is a direct product.)

Let $G = AB$ a factorized group:

$$A, B \in \mathcal{U} \text{ (or } \mathcal{F}\text{)} + \boxed{\text{permutability properties}} \implies G \in \mathcal{U} \text{ (or } \mathcal{F}\text{)}$$

Total permutability

Definition

Let G be a group and let A and B be subgroups of G. It is said that A and B are totally permutable if every subgroup of A permutes with every subgroup of B.

Theorem

(Asaad,Shaalan, 89) *If $G = AB$ is the product of the totally permutable subgroups A and B, then*

$$A, B \in \mathcal{U} \implies G \in \mathcal{U}$$

Total permutability and formations

(Maier,92; Carocca,96;Ballester-Bolinches, Pedraza-Aguilera, Pérez-Ramos, 96-98) Let $\mathcal{F}$ be a formation such that $\mathcal{U} \subseteq \mathcal{F}$. Let the group $G = G_1 G_2 \cdots G_r$ be a product of pairwise totally permutable subgroups $G_1, G_2, \ldots, G_r$, $r \geq 2$. Then:

Theorem

- *If $G_i \in \mathcal{F}$ $\forall i \in \{1, \ldots, r\}$, then $G \in \mathcal{F}$.*
- *Assume in addition that $\mathcal{F}$ is either saturated or $\mathcal{F} \subseteq \mathcal{S}$. If $G \in \mathcal{F}$, then $G_i \in \mathcal{F}$, $\forall i \in \{1, \ldots, r\}$.*

Corollary

- *If $\mathcal{F}$ is either saturated or $\mathcal{F} \subseteq \mathcal{S}$, then: $G^{\mathcal{F}} = G_1^{\mathcal{F}} G_2^{\mathcal{F}} \ldots G_r^{\mathcal{F}}$.*

Conditional permutability

Definitions

(Qian,Zhu,98) (Guo, Shum, Skiba, 05) Let G be a group and let A and B be subgroups of G.

- A and B are conditionally permutable in G (c-permutable), if $AB^g = B^gA$ for some $g \in G$.
- A and B are totally c-permutable if every subgroup of A is c-permutable in G with every subgroup of B.

$$\text{Permutable} \;\; \underset{\not\Longleftarrow}{\Longrightarrow} \;\; \text{Conditionally Permutable}$$

Example

Let X and Y be two 2-Sylow subgroups of S_3. Then X permutes with Y^g for some $g \in S_3$, but X does not permute with Y.

Total c-permutability and supersolubility

Theorem

(Arroyo-Jordá, AJ, Martínez-Pastor,Pérez-Ramos,10) *Let $G = AB$ be the product of the totally c-permutable subgroups A and B. Then:*

$$G^{\mathcal{U}} = A^{\mathcal{U}} B^{\mathcal{U}}$$

Total c-permutability and supersolubility

Theorem

(Arroyo-Jordá, AJ, Martínez-Pastor,Pérez-Ramos,10) *Let $G = AB$ be the product of the totally c-permutable subgroups A and B. Then:*

$$G^{\mathcal{U}} = A^{\mathcal{U}} B^{\mathcal{U}}$$

In particular, $$A, B \in \mathcal{U} \iff G \in \mathcal{U}$$

Total c-permutability and supersolubility

Theorem

(Arroyo-Jordá, AJ, Martínez-Pastor,Pérez-Ramos,10) *Let $G = AB$ be the product of the totally c-permutable subgroups A and B. Then:*

$$G^{\mathcal{U}} = A^{\mathcal{U}} B^{\mathcal{U}}$$

In particular, $$A, B \in \mathcal{U} \iff G \in \mathcal{U}$$

Corollary

(AJ, AJ, MP, PR, 10) *Let $G = AB$ be the product of the totally c-permutable subgroups A and B and let p be a prime. If A, B are p-supersoluble, then G is p-supersoluble.*

Total c-permutability and saturated formations

Question

Are saturated formations $\mathcal{F}$ (of soluble groups) containing $\mathcal{U}$ closed under taking products of totally c-permutable subgroups?

Example

Take $G = S_4 = AB$, $A = A_4$ and $B \cong C_2$ generated by a transposition. Then A and B are totally c-permutable in G.

Let $\mathcal{F} = \mathcal{N}^2$, the saturated formation of metanilpotent groups. Notice $\mathcal{U} \subseteq \mathcal{N}^2 \subseteq S$. Then:

$$A, B \in \mathcal{F} \text{ but } G \notin \mathcal{F}.$$

In particular, $G^{\mathcal{F}} \neq A^{\mathcal{F}} B^{\mathcal{F}}$.

Conditional permutability

Remark

c-permutability fails to satisfy the property of persistence in intermediate subgroups.

Example

Let $G = S_4$ and let $Y \cong C_2$ generated by a transposition.

Let V be the normal subgroup of G of order 4 and X a subgroup of V of order 2, $X \neq Z(VY)$. Then

X and Y are c-permutable in G

X and Y are not c-permutable in $\langle X, Y\rangle$.

Complete c-permutability

Definitions

(Guo, Shum, Skiba, 05) Let G be a group and let A and B be subgroups of G.

- A and B are completely c-permutable in G (cc-permutable), if $AB^g = B^g A$ for some $g \in \langle A, B \rangle$.
- A and B are totally completely c-permutable (tcc-permutable) if every subgroup of A is completely c-permutable in G with every subgroup of B.

Totally permutable $\underset{\not\Longleftarrow}{\Longrightarrow}$ Totally completely c-permutable $\underset{\not\Longleftarrow}{\Longrightarrow}$ Totally c-permutable

Complete c-permutability and supersolubility

$$G = AB, \;\; A, B \text{ totally c-permutable}, \; G^{\mathcal{U}} = A^{\mathcal{U}} B^{\mathcal{U}}$$

Corollary

(Guo, Shum, Skiba, 06)

- *Let $G = AB$ be a product of the tcc-permutable subgroups A and B. If $A, B \in \mathcal{U}$, then $G \in \mathcal{U}$.*

Complete c-permutability and supersolubility

$G = AB$, A, B totally c-permutable, $G^{\mathcal{U}} = A^{\mathcal{U}} B^{\mathcal{U}}$

Corollary

(Guo, Shum, Skiba, 06)

- *Let $G = AB$ be a product of the tcc-permutable subgroups A and B. If $A, B \in \mathcal{U}$, then $G \in \mathcal{U}$.*
- *Let $G = AB$ be the product of the tcc-permutable subgroups A and B and let p be a prime. If A, B are p-supersoluble, then G is p-supersoluble.*

Total complete c-permutability and saturated formations

Question

Are saturated formations $\mathcal{F}$ (of soluble groups) containing $\mathcal{U}$ closed under taking products of totally completely c-permutable subgroups?

Theorem

(Arroyo-Jordá, AJ, Pérez-Ramos, 11)
Let $\mathcal{F}$ be a saturated formation such that $\mathcal{U} \subseteq \mathcal{F} \subseteq \mathcal{S}$.
Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then:

- *If $G_i \in \mathcal{F}$ for all $i = 1, \ldots, r$, then $G \in \mathcal{F}$.*
- *If $G \in \mathcal{F}$, then $G_i \in \mathcal{F}$ for all $i = 1, \ldots, r$.*

Total complete c-permutability and saturated formations

Corollary

(Arroyo-Jordá, AJ, Pérez-Ramos, 11)
Let $\mathcal{F}$ be a saturated formation such that $\mathcal{U} \subseteq \mathcal{F} \subseteq \mathcal{S}$. Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \dots, G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \dots, r\}$, $i \neq j$. Then:

- $G_i^{\mathcal{F}} \trianglelefteq G$ for all $i = 1, \dots, r$.
- $G^{\mathcal{F}} = G_1^{\mathcal{F}} \cdots G_r^{\mathcal{F}}$.

Total complete c-permutability and saturated formations

Question

Is it possible to extend the above results on products of tcc-permutable subgroups to either non-saturated formations or saturated formations of non-soluble groups $\mathcal{F}$ such that $\mathcal{U} \subseteq \mathcal{F}$?

Total complete c-permutability and saturated formations

Question

Is it possible to extend the above results on products of tcc-permutable subgroups to either non-saturated formations or saturated formations of non-soluble groups $\mathcal{F}$ such that $\mathcal{U} \subseteq \mathcal{F}$?

- We need a better knowledge of structural properties of products of tcc-permutable groups.

Structural properties

Lemma

(AJ, AJ, PR, 11) *If $1 \neq G = AB$ is the product of tcc-permutable subgroups A and B, then there exists $1 \neq N \trianglelefteq G$ such that either $N \leq A$ or $N \leq B$.*

Corollary

(AJ, AJ, PR,11) *Let the group $1 \neq G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$.*

Then there exists $1 \neq N \trianglelefteq G$ such that $N \leq G_i$ for some $i \in \{1, \ldots, r\}$.

Subnormal subgroups

Proposition

(AJ,AJ,MP,PR,13) *Let the group $G = AB$ be the product of tcc-permutable subgroups A and B. Then*

$$A' \trianglelefteq \trianglelefteq G \text{ and } B' \trianglelefteq \trianglelefteq G.$$

Corollary

(AJ,AJ,MP,PR,13) *Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \dots, G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \dots, r\}$, $i \neq j$. Then:*

$$G_i' \trianglelefteq \trianglelefteq G, \quad \text{for all } i \in \{1, \dots, r\}.$$

Subnormal subgroups

Proposition

(Maier, 92) *If $G = AB$ is the product of totally permutable subgroups A and B, then $A \cap B \leq F(G)$, that is, $A \cap B$ is a subnormal nilpotent subgroup of G*

Example

The above property is not true for products of tcc-permutable subgroups.

- Let $G = S_3 = AB$ with the trivial factorization $A = S_3$ and B a 2-Sylow subgroup of G. This is a product of tcc-permutable subgroups, but: $A \cap B = B$ is not a subnormal subgroup of G.
- Let $G = S_3 = AB$ with the trivial factorization $A = B = S_3$. This is a product of tcc-permutable subgroups, but: $A \cap B = S_3 \notin \mathcal{N}$.

Nilpotent residuals

Theorem

(Beidleman, Heineken, 99) *Let $G = AB$ be a product of the totally permutable subgroups A and B. Then:*

$$[A^{\mathcal{N}}, B] = 1 \text{ and } [B^{\mathcal{N}}, A] = 1.$$

Example

Let $V = \langle a, b\rangle \cong C_5 \times C_5$ and $C_6 \cong C = \langle \alpha, \beta\rangle \leq \mathrm{Aut}(V)$ given by:

$$a^\alpha = a^{-1},\ b^\alpha = b^{-1};\ a^\beta = b,\ b^\beta = a^{-1}b^{-1}$$

Let $G = [V]C$ be the corresponding semidirect product. Then $G = AB$ is the product of the tcc-permutable subgroups $A = \langle\alpha\rangle$ and $B = V\langle\beta\rangle$. Notice that $A \in \mathcal{U}$, but

$$B^{\mathcal{N}} = B^{\mathcal{U}} = V \text{ does not centralize } A.$$

Nilpotent residuals

Theorem

(AJ,AJ,MP,PR,13) *Let the group $G = AB$ be the product of tcc-permutable subgroups A and B. Then*

$$A^{\mathcal{N}} \trianglelefteq G \text{ and } B^{\mathcal{N}} \trianglelefteq G.$$

Corollary

(AJ,AJ,MP,PR,13) *Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then*

$$G_i^{\mathcal{N}} \trianglelefteq G, \quad \text{for all } i \in \{1, \ldots, r\}.$$

$\mathcal{U}$-hypercentre

Theorem

(Hauck, PR, MP, 03), (Gállego, Hauck, PR, 08) *Let $G = AB$ be a product of the totally permutable subgroups A and B. Then:*

$$[A, B] \leq Z_{\mathcal{U}}(G)$$

or, equivalently, $G/Z_{\mathcal{U}}(G) = AZ_{\mathcal{U}}(G)/Z_{\mathcal{U}}(G) \times BZ_{\mathcal{U}}(G)/Z_{\mathcal{U}}(G)$.

Example

Let $G = [V]C = AB$ the product of the tcc-permutable subgroups $A = \langle \alpha \rangle$ and $B = V\langle \beta \rangle$ (under the action $a^{\alpha} = a^{-1}$, $b^{\alpha} = b^{-1}$; $a^{\beta} = b$, $b^{\beta} = a^{-1}b^{-1}$). Notice that:

$$Z_{\mathcal{U}}(G) = 1 \text{ and } G \text{ is not a direct product of } A \text{ and } B.$$

Main theorem

Theorem

(AJ,AJ,MP,PR, 13)

Let the group $G = AB$ be the product of tcc-permutable subgroups A and B. Then:

$$[A, B] \leq F(G).$$

- For the proof we have used the CFSG.

Consequences of the main theorem

Corollary

(AJ,AJ,MP,PR, 13) *Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \dots, G_r$, for $r \geq 2$, and $G_i \neq 1$ for all $i = 1, \dots, r$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \dots, r\}$, $i \neq j$. Let N be a minimal normal subgroup of G. Then:*

1. *If N is non-abelian, then there exists a unique $i \in \{1, \dots, r\}$ such that $N \leq G_i$. Moreover, G_j centralizes N and $N \cap G_j = 1$, for all $j \in \{1, \dots, r\}$, $j \neq i$.*

Consequences of the main theorem

Corollary

(AJ,AJ,MP,PR, 13) *Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \dots, G_r$, for $r \geq 2$, and $G_i \neq 1$ for all $i = 1, \dots, r$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \dots, r\}$, $i \neq j$. Let N be a minimal normal subgroup of G. Then:*

1. *If N is non-abelian, then there exists a unique $i \in \{1, \dots, r\}$ such that $N \leq G_i$. Moreover, G_j centralizes N and $N \cap G_j = 1$, for all $j \in \{1, \dots, r\}$, $j \neq i$.*
2. *If G is a monolithic primitive group, then the unique minimal normal subgroup N is abelian.*

Consequences of the main theorem

Corollary

(AJ,AJ,MP,PR, 13) *Let the group $G = AB$ be the tcc-permutable product of the subgroups A and B. Then:*

- *If A is a normal subgroup of G, then B acts u-hypercentrally on A by conjugation. In particular, $B^{\mathcal{U}}$ centralizes A.*
- $[A^{\mathcal{U}}, B^{\mathcal{U}}] = 1$.

Total complete c-permutability and formations

Theorem

(AJ, AJ, MP, PR, 13) *Let $\mathcal{F}$ be a saturated formation such that $\mathcal{U} \subseteq \mathcal{F}$. Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \ldots, G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then:*

- *If $G_i \in \mathcal{F}$ for all $i = 1, \ldots, r$, then $G \in \mathcal{F}$.*
- *If $G \in \mathcal{F}$, then $G_i \in \mathcal{F}$ for all $i = 1, \ldots, r$.*

Total complete c-permutability and formations

Theorem

(AJ, AJ, MP, PR, 13) *Let $\mathcal{F}$ be a saturated formation such that $\mathcal{U} \subseteq \mathcal{F}$. Let the group $G = G_1 \cdots G_r$ be the product of pairwise permutable subgroups $G_1, \dots, G_r$, for $r \geq 2$. Assume that G_i and G_j are tcc-permutable subgroups for all $i, j \in \{1, \dots, r\}$, $i \neq j$. Then:*

- *If $G_i \in \mathcal{F}$ for all $i = 1, \dots, r$, then $G \in \mathcal{F}$.*
- *If $G \in \mathcal{F}$, then $G_i \in \mathcal{F}$ for all $i = 1, \dots, r$.*

Corollary

Under the same hypotheses:

- $G_i^{\mathcal{F}} \trianglelefteq G$ *for all* $i = 1, \dots, r$.
- $G^{\mathcal{F}} = G_1^{\mathcal{F}} \cdots G_r^{\mathcal{F}}$.

Necessity of saturation

Example

Define the mapping $f : \mathbb{P} \longrightarrow \{$ classes of groups $\}$ by setting

$$f(p) = \begin{cases} (1, C_2, C_3, C_4) & \text{if } p = 5 \\ (G \in \mathcal{A} : exp(G) \mid p-1) & \text{if } p \neq 5 \end{cases}$$

Let $\mathcal{F} = (G \in \mathcal{S} \mid H/K$ chief factor of $G \Rightarrow \mathrm{Aut}_G(H/K) \in f(p)\ \forall p \in \sigma(H/K))$.

$\mathcal{F}$ is a formation of soluble groups such that $\mathcal{U} \subseteq \mathcal{F}$.

Let again $G = [V]C = AB$ be the product of the tcc-permutable subgroups $A = \langle\alpha\rangle$ and $B = V\langle\beta\rangle$ (under the action $a^\alpha = a^{-1}$, $b^\alpha = b^{-1}$; $a^\beta = b$, $b^\beta = a^{-1}b^{-1}$). Then:

- $A, B \in \mathcal{F}$, but $G \notin \mathcal{F}$, since $G/C_G(V) \cong C_3 \times C_2 \notin f(5)$.

Necessity of saturation

Example

Define the mapping $f : \mathbb{P} \longrightarrow \{$ classes of groups $\}$ by setting

$$f(p) = \begin{cases} (1, C_2, C_3, C_4) & \text{if } p = 5 \\ (G \in \mathcal{A} : exp(G) \mid p-1) & \text{if } p \neq 5 \end{cases}$$

Let $\mathcal{F} = (G \in \mathcal{S} \mid H/K \text{ chief factor of } G \Rightarrow \mathrm{Aut}_G(H/K) \in f(p)\ \forall p \in \sigma(H/K))$.

$\mathcal{F}$ is a formation of soluble groups such that $\mathcal{U} \subseteq \mathcal{F}$.

Let again $G = [V]C = AB$ be the product of the tcc-permutable subgroups $A = \langle\alpha\rangle$ and $B = V\langle\beta\rangle$ (under the action $a^\alpha = a^{-1}$, $b^\alpha = b^{-1}$; $a^\beta = b$, $b^\beta = a^{-1}b^{-1}$). Then:

- $A, B \in \mathcal{F}$, but $G \notin \mathcal{F}$, since $G/C_G(V) \cong C_3 \times C_2 \notin f(5)$.

Modifying the construction of the formation $\mathcal{F}$ by setting $f(5) = (1, C_2, C_4, C_6)$:

- $G, A \in \mathcal{F}$, but $B \notin \mathcal{F}$, since $B/C_B(V) \cong C_3 \notin f(5)$.

References

M. Arroyo-Jordá, P. Arroyo-Jordá, A. Martínez-Pastor and M. D. Pérez-Ramos, On finite products of groups and supersolubility, *J. Algebra*, 323 (2010), 2922-2934.

M. Arroyo-Jordá, P. Arroyo-Jordá and M. D. Pérez-Ramos, On conditional permutability and saturated formations, *Proc. Edinburgh Math. Soc.*, 54 (2011), 309-319.

M. Arroyo-Jordá, P. Arroyo-Jordá, A. Martínez-Pastor and M. D. Pérez-Ramos, A survey on some permutability properties on subgroups of finite groups, *Proc. Meeting on group theory and its applications, on the occasion of Javier Otal's 60 th birthday*, Biblioteca RMI, (2012), 1-11.

M. Arroyo-Jordá, P. Arroyo-Jordá, A. Martínez-Pastor and M. D. Pérez-Ramos, On conditional permutability and factorized groups, *Annali di Matematica Pura ed Applicata* (2013), DOI 10.1007/S10231-012-0319-1.

THANK YOU FOR YOUR ATTENTION!