Vrije Universiteit Brussel

Rational conjugacy of torsion units in integral group rings of non-solvable groups

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$\mathrm{U}(RG) = R^\times \cdot \mathrm{V}(RG)$

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Prime graph question (W. Kimmerle, 2006)

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Clearly: **(ZC1)** $\Rightarrow$ **(PQ)**.

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- ✓ $\mathrm{PSL}(2,8)$, $\mathrm{PSL}(2,17)$ (Gildea; Kimmerle, Konovalov 2012)

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- ✓ solvable groups (Höfert, Kimmerle, 2006)
- ✓ $\mathrm{PSL}(2, p)$, p a rational prime (Hertweck 2007)
- ✓ certain sporadic simple groups (Bovdi, Konovalov, et. al. 2005 –)

Theorem (Kimmerle, Konovalov 2012)

(PQ) *holds for all groups, whose order is divisible by at most three primes, if there are no units of order* 6 *in* $\mathrm{V}(\mathbb{Z}\,\mathrm{PGL}(2,9))$ *and in* $\mathrm{V}(\mathbb{Z}M_{10})$.

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If the order of a group is devisible by at most three different rational primes, then **(PQ)** *holds for this groups.*

Theorem (Bächle, Margolis 2013)

(ZC1) *holds for* $\mathrm{PSL}(2,19)$ *and* $\mathrm{PSL}(2,23)$.

HeLP 1

Let $x \in G$, x^G its conjugacy class in G, and
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Lemma (Marciniak, Ritter, Sehgal, Weiss 1987; Luthar, Passi 1989)

Let $u \in \mathrm{V}(\mathbb{Z}G)$ be of finite order. Then u is conjugate to an element of G in $\mathbb{Q}G \Leftrightarrow \varepsilon_g(u) \geq 0$ for every $g \in G$.

HeLP 2

Theorem (Luthar, Passi, 1989; Hertweck, 2004)

- *$u \in \mathbb{Z}G$ torsion unit of order n*
- *F splitting field for G with $\mathrm{char}(F) \nmid n$*
- *χ a (Brauer) character of F-representation D of G*
- *$\zeta \in \mathbb{C}$ primitive n-th root of unity*
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Multiplicity of ξ^{ℓ} as an eigenvalue of $D(u)$ is given by

$$\frac{1}{n}\sum_{\substack{d|n \\ d\neq 1}} \mathrm{Tr}_{\mathbb{Q}(\zeta^d)/\mathbb{Q}}(\chi(u^d)\zeta^{-d\ell}) + \frac{1}{n}\sum_{x^G} \mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi(x)\zeta^{-\ell})\varepsilon_x(u)$$

Example $G = A_6$

	1a	2a	3a	3b	4a	5a	5b
χ	5	1	2	-1	...	...	...

Assume: $u \in V(\mathbb{Z}A_6)$ has order 6, u^4 is rationally conjugate to an element of 3b and u^3 is rationally conjugate to an element of 2a, $\varepsilon_{2a}(u) = -2$, $\varepsilon_{3a}(u) = 2$, $\varepsilon_{3b}(u) = 1$.

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If D affords χ and $\zeta \in \mathbb{C}$ is a primitive 3rd root of unity,

$$D(u^3) \sim \mathrm{diag}(1, 1, 1, -1, -1), \quad D(u^4) \sim \mathrm{diag}(1, \zeta, \zeta^2, \zeta, \zeta^2)$$

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Hence, as

$$\chi(u) = \varepsilon_{2a}(u)\chi(2a) + \varepsilon_{3a}(u)\chi(3a) + \varepsilon_{3b}(u)\chi(3b) = 1,$$

there is only the possibility

$$D(u) \sim \mathrm{diag}(1, \zeta, \zeta^2, -\zeta, -\zeta^2).$$

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- $u \in \mathrm{V}(\mathbb{Z}G)$ a torsion unit
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- K the p-adic completation of a number field admitting D with minimal ramification index over $\mathbb{Q}_p$
- R the ring of interges of K with maximal ideal P containing p
- L an RG-lattice affording D
- $k = R/P$, the quotient field, and $\bar{\ }$ the reduction mod P

Proposition

Let $o(u) = p^a m$, $p \nmid m$. *Let* $\zeta \in R$ *be a primitive* m*-th root of unity. Let* A_j *be tuples of* p^a*-th roots of unity s.t. the eigenvalues of* $D(u)$ *are* $\zeta A_1 \cup \zeta^2 A_2 \cup \ldots \cup \zeta^m A_m$. *Then, as* $R\langle u\rangle$*-lattice,* $L \simeq M_1 \oplus \ldots \oplus M_m$ *where* $\mathrm{rank}_R(M_j) = |A_j| = \dim_k(\bar{M}_j)$ *and* $\bar{M}_j$ *has only one composition factor up to isomorphism.*

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Easiest case: $K/\mathbb{Q}_p$ unramified, $o(u) = p$.
Precisely three indecomposable $R\langle u\rangle$-lattices: R, $I(RC_p)$, RC_p of rank 1, $p-1$, p, respectively, with corresponding eigenvalues $\{1\}$, $\{\xi, \ldots, \xi^{p-1}\}$, $\{1, \xi, \ldots, \xi^{p-1}\}$, where ξ is a primitive p-th root of unity. The reduction of any such lattice modulo P stays indecomposable.

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elements $u \in \mathrm{V}(\mathbb{Z}G)$ of order 10 having partial augmentations

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Let ζ be a 5th primitive root of unity, D_{18}, D_{19} (certain) ordinary representations of G, D_{18} can be realized over $\mathbb{Z}_5[\zeta + \zeta^{-1}]$, D_{19} can be realized over $\mathbb{Z}_5$. Let L_{18} and L_{19} be corresponding RG-lattices (note that the R's are different), then $\bar{L}_{18} \leq \bar{L}_{19}$ and $\bar{L}_{19}/\bar{L}_{18}$ is a trivial kG-module.

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Using partial augmentations:

$$D_{18}(u) \sim \mathrm{diag}(\overbrace{-1, -\zeta, -\zeta^2, -\zeta^3, -\zeta^4}^{\text{5th roots of unity}}, -1, -\zeta, -\zeta^4, -\zeta, -\zeta^4)$$

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$$\bar{L}_{18} \simeq M_{18}^{1} \oplus M_{18}^{-1}, \qquad \bar{L}_{19} \simeq M_{19}^{1} \oplus M_{19}^{-1}$$

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$$M_{19}^{-1} \in \{2(k)_- \oplus 2I(kC_5)_-,\ (k)_- \oplus I(kC_5)_- \oplus (kC_5)_-,\ 2(kC_5)_-\}.$$

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As $\bar{L}_{19}/\bar{L}_{18}$ is a trivial kG-module, we have $M_{19}^{-1} \simeq M_{18}^{-1}$. But this is impossible as the composition factors of M_{18}^{-1} as $k\langle \bar{u} \rangle$-module can't coincide with those of M_{19}^{-1} we just calctulated (using results of Gudivok (1965) and Jacobinski (1967)).

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Thank you for your attention!