On Clifford-Fischer Theory

Ayoub Basheer*
School of Mathematics, Statistics & Computer Science, University of KwaZulu-Natal, Pietermaritzburg
Department of Mathematics, Faculty of Mathematical Sciences, University of Khartoum, P. O. Box 321, Khartoum, Sudan

Jamshid Moori
School of Mathematical Sciences, North-West University, Mafikeng

Groups St Andrews 2013 in University of St Andrews, Scotland
3rd-11th of August 2013

Abstract

Bernd Fischer presented a powerful and interesting technique, known as *Clifford-Fischer theory*, for calculating the character tables of group extensions. This technique derives its fundamentals from the Clifford theory. In this talk we describe the methods of the coset analysis and Clifford-Fischer theory applied to group extensions (split and non-split). We also mention some of the contributions to this domain and in particular of the second author and his research groups including students.

The Character Table of a Group Extension

- Let $\overline{G} = N \cdot G$, where $N \lhd \overline{G}$ and $\overline{G}/N \cong G$, be a finite group extension.
- There are several well-developed methods for calculating the character tables of group extensions. For example, the Schreier-Sims algorithm, the Todd-Coxeter coset enumeration method, the Burnside-Dixon algorithm and various other techniques.
- Bernd Fischer [11, 12, 13] presented a powerful and interesting technique, known nowadays as the *Clifford-Fischer Theory*, for calculating the character tables of group extensions. To construct the character table of $\overline{G}$ using this method, we need to have
  1. the conjugacy classes of $\overline{G}$ obtained through the coset analysis method,
  2. the character tables (ordinary or projective) of the inertia factor groups,
  3. the fusions of classes of the inertia factors into classes of G,
  4. the Fischer matrices of $\overline{G}$.

Coset Analysis Technique

- For each $g \in G$ let $\overline{g} \in \overline{G}$ map to g under the natural epimorphism $\pi : \overline{G} \longrightarrow G$ and let $g_1 = N\overline{g}_1, g_2 = N\overline{g}_2, \cdots, g_r = N\overline{g}_r$ be representatives for the conjugacy classes of $G \cong \overline{G}/N$. Therefore $\overline{g}_i \in \overline{G}, \ \forall i$, and by convention we take $\overline{g}_1 = 1_{\overline{G}}$.
- The method of the coset analysis constructs for each conjugacy class $[g_i]_G, \ 1 \leq i \leq r$, a number of conjugacy classes of $\overline{G}$. For each $1 \leq i \leq r$, we let $g_{i1}, g_{i2}, \cdots, g_{ic(g_i)}$ be the corresponding representatives of these classes. That is each conjugacy class of $\overline{G}$ corresponds uniquely to a conjugacy class of G.
- Also we use the notation $U = \pi(\overline{U})$ for any subset $\overline{U} \subseteq \overline{G}$. Thus we have $\pi^{-1}([g_i]_G) = \bigcup_{j=1}^{c(g_i)} [g_{ij}]_{\overline{G}}$ for any $1 \leq i \leq r$. We assume that $\pi(g_{ij}) = g_i$ and by convention we may take $g_{11} = 1_{\overline{G}}$.

Coset Analysis Technique

The coset analysis method can be described briefly in the following steps:

- For fixed $i \in \{1, 2, \cdots, r\}$, act N (by conjugation) on the coset $N\overline{g}_i$ and let the resulting orbits be $Q_{i1}, Q_{i2}, \cdots, Q_{ik_i}$. If N is abelian (regardless to whether the extension is split or not), then $|Q_{i1}| = |Q_{i2}| = \cdots = |Q_{ik_i}| = \frac{|N|}{k_i}$.
- Act $\overline{G}$ on $Q_{i1}, Q_{i2}, \cdots, Q_{ik_i}$ and suppose f_{ij} orbits fuse together to form a new orbit Δ_{ij}. Let the total number of the new resulting orbits in this action be $c(g_i)$ (that is $1 \leq j \leq c(g_i)$). Then $\overline{G}$ has a conjugacy class $[g_{ij}]_{\overline{G}}$ that contains Δ_{ij} and $|[g_{ij}]_{\overline{G}}| = |[g_i]_G| \times |\Delta_{ij}|$.
- Repeat the above two steps, for all $i \in \{1, 2, \cdots, r\}$.

Example of Using the Coset Analysis Technique

- In [10] we used the coset analysis to compute the conjugacy classes of $\overline{G} = 2_{-}^{1+6}{:}((3^{1+2}{:}8){:}2)$. This is a maximal subgroup, of index 3, in $2_{-}^{1+6}{:}3_{-}^{1+2}{:}2S_4$, which in turn is the second largest maximal subgroup of the automorphism group of the unitary group $U_5(2)$.
- Using the coset analysis we found that corresponding to the 14 classes of $G = (3^{1+2}{:}8){:}2$, we obtain 41 conjugacy classes for $\overline{G}$. For example the group G has two classes of involutions represented by 2_1 and 2_2 with respective centralizer sizes 48 and 12. Corresponding to the class containing 2_2 we get five conjugacy classes in $\overline{G}$ with information listed in the following table.

$[g_i]_G$	k_i	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
$g_3 = 2_2$	$k_3 = 9$	$m_{31} = 8$	g_{31}	8	288	192
		$m_{32} = 8$	g_{32}	8	288	192
		$m_{33} = 24$	g_{33}	2	576	96
		$m_{34} = 48$	g_{34}	8	1728	32
		$m_{35} = 48$	g_{35}	4	1728	32

Inertia Factor Groups

- If $\overline{G} = N \cdot G$ is a group extension, then $\overline{G}$ has action on the classes of N and also on $\mathrm{Irr}(N)$. Brauer Theorem (see [3] for example) asserts that the number of orbits of these two actions are the same.
- Let $\theta_1, \theta_2, \cdots, \theta_t$ be representatives of $\overline{G}-$orbits on $\mathrm{Irr}(N)$ and let $\overline{H}_k$ and H_k denote the corresponding inertia and inertia factor groups of θ_k.
- In order to apply the Clifford-Fischer Theory, one have to determine the structures of all the inertia or inertia factor groups.
- The Clifford Theory (see [3]) deals with the character tables (ordinary or projective) of the inertia groups.

Inertia Factor Groups

- In practise we do not attempt to compute the character table of $\overline{H}_k$, simply because the character tables of these inertia groups are usually much larger and more complicated to compute than the character table of $\overline{G}$ itself.
- Bernd Fischer suggested to use the character tables of the inertia factor groups H_k together with some matrices, called by him *Clifford matrices* (throughout this talk we refer to them as *Fischer matrices*), to construct the character table of $\overline{G}$.
- Thus we firstly need to determine the structures and the appropriate projective character table of all the inertia factors H_k together with the Fischer matrices.
- One of the biggest challenges in Clifford-Fischer theory is the determination of the type of the character table of H_k (projective or ordinary), which is to be used in the construction of the character table of $\overline{G}$.

Inertia Factor Groups

- In practice making the right choice of the appropriate projective character table of H_k, with factor set α_k, might be difficult unless the Schur multipliers of all the H_k are trivial.
- Otherwise there will be many combinations (for each H_k, there are many projective character tables associated with different factor sets of the Schur multiplier of H_k) and one has to test all the possible choices and eliminate the choices that lead to contradictions.
- Some partial results on the extendability of characters are given in [3].
- Having determined the structures and the appropriate projective character table of H_k, with factor set α_k (that is to be used to construct the character table of $\overline{G}$), the next step will be to determine the fusions of the α_k-regular classes of H_k into classes of G.

Some Notations & Settings

- We proceed to define the Fischer matrices, which are so important to calculate the character table of any group extension $\overline{G} = N \cdot G,\ N \lhd \overline{G}$.
- For each $[g_i]_G$, there corresponds a Fischer matrix $\mathcal{F}_i$.
- $[g_{ij}]_{\overline{G}} \bigcap \overline{H}_k = \bigcup_{n=1}^{c(g_{ijk})} [g_{ijkn}]_{\overline{H}_k}$, where $g_{ijkn} \in \overline{H}_k$ and by $c(g_{ijk})$ we mean the number of $\overline{H}_k-$conjugacy classes that form a partition for $[g_{ij}]_{\overline{G}}$. Since $g_{11} = 1_{\overline{G}}$, we have $g_{11k1} = 1_{\overline{G}}$ and thus $c(g_{11k1}) = 1$ for all $1 \leq k \leq t$
- $[g_i]_G \bigcap H_k = \bigcup_{m=1}^{c(g_{ik})} [g_{ikm}]_{H_k}$, where $g_{ikm} \in H_k$ and by $c(g_{ik})$ we mean the number of H_k-conjugacy classes that form a partition for $[g_i]_G$. Since $g_1 = 1_G$, we have $g_{1k1} = 1_G$ and thus $c(g_{1k1}) = 1$ for all $1 \leq k \leq t$. Also $\pi(g_{ijkn}) = g_{ikm}$ for some $m = f(j, n)$.

Labeling Columns & Rows of Fischer Matrices

- The top of the columns of $\mathcal{F}_i$ are labeled by the representatives of $[g_{ij}]_{\overline{G}}$, $1 \leq j \leq c(g_i)$ obtained by the coset analysis and below each g_{ij} we put $|C_{\overline{G}}(g_{ij})|$.
- The bottom of the columns of $\mathcal{F}_i$ are labeled by some weights m_{ij} defined by

$$m_{ij} = [N_{\overline{G}}(N\overline{g}_i) : C_{\overline{G}}(g_{ij})] = |N|\frac{|C_G(g_i)|}{|C_{\overline{G}}(g_{ij})|}.$$

- To label the rows of $\mathcal{F}_i$ we define the set J_i to be

$$J_i = \{(k,m)|\ 1 \leq k \leq t,\ 1 \leq m \leq c(g_{ik}),\ g_{ikm} \text{ is } \alpha_k^{-1} - \text{regular class}\}.$$

- Then each row of $\mathcal{F}_i$ is indexed by a pair $(k,m) \in J_i$.

The Fischer Matrix $\mathcal{F}_i$ Corresponds to $[g_i]_G$

- For fixed $1 \leq k \leq t$, we let $\mathcal{F}_{ik}$ be a sub-matrix of $\mathcal{F}_i$ with rows correspond to the pairs $(k,1),(k,2),\cdots,(k,r_k)$.
- Let

$$a_{ij}^{(k,m)} := \sum_{n=1}^{c(g_{ijk})} \frac{|C_{\overline{G}}(g_{ij})|}{|C_{\overline{H}_k}(g_{ijkn})|} \widetilde{\psi}_k(g_{ijkn})$$

(for which $\pi(g_{ijkn}) = g_{ikm}$).

- For each i, corresponding to the conjugacy class $[g_i]_G$, we define the Fischer matrix $\mathcal{F}_i = \left(a_{ij}^{(k,m)}\right)$, where $1 \leq k \leq t,\ 1 \leq m \leq c(g_{ik}),\ 1 \leq j \leq c(g_i)$.

The Fischer Matrix $\mathcal{F}_i$ Corresponds to $[g_i]_G$

- The Fischer matrix $\mathcal{F}_i$,

$$\mathcal{F}_i = \left(a_{ij}^{(k,m)}\right) = \left(\begin{array}{c} \mathcal{F}_{i1} \\ \hline \mathcal{F}_{i2} \\ \hline \vdots \\ \hline \mathcal{F}_{it} \end{array}\right)$$

together with additional information required for their definition are presented as follows:

The Fischer Matrix $\mathcal{F}_i$ With Some Additional Information

$\mathcal{F}_i$

g_i		g_{i1}	g_{i2}	$\cdots$	$g_{ic(g_i)}$
$\lvert C_{\overline{G}}(g_{ij})\rvert$		$\lvert C_{\overline{G}}(g_{i1})\rvert$	$\lvert C_{\overline{G}}(g_{i2})\rvert$	$\cdots$	$\lvert C_{\overline{G}}(g_{ic(g_i)})\rvert$
(k,m)	$\lvert C_{H_k}(g_{ikm})\rvert$				
$(1,1)$	$\lvert C_G(g_i)\rvert$	$a_{i1}^{(1,1)}$	$a_{i2}^{(1,1)}$	$\cdots$	$a_{ic(g_i)}^{(1,1)}$
$(2,1)$	$\lvert C_{H_2}(g_{i21})\rvert$	$a_{i1}^{(2,1)}$	$a_{i2}^{(2,1)}$	$\cdots$	$a_{ic(g_i)}^{(2,1)}$
$(2,2)$	$\lvert C_{H_2}(g_{i22})\rvert$	$a_{i1}^{(2,2)}$	$a_{i2}^{(2,2)}$	$\cdots$	$a_{ic(g_i)}^{(2,2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(2,r_2)$	$\lvert C_{H_2}(g_{i2r_{i2}})\rvert$	$a_{i1}^{(2,r_2)}$	$a_{i2}^{(2,r_2)}$	$\cdots$	$a_{ic(g_i)}^{(2,r_2)}$
$(u,1)$	$\lvert C_{H_u}(g_{iu1})\rvert$	$a_{i1}^{(u,1)}$	$a_{i2}^{(u,1)}$	$\cdots$	$a_{ic(g_i)}^{(u,1)}$
$(u,2)$	$\lvert C_{H_u}(g_{iu2})\rvert$	$a_{i1}^{(u,2)}$	$a_{i2}^{(u,2)}$	$\cdots$	$a_{ic(g_i)}^{(u,2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
(u,r_u)	$\lvert C_{H_u}(g_{iur_{iu}})\rvert$	$a_{i1}^{(u,r_u)}$	$a_{i2}^{(u,r_u)}$	$\cdots$	$a_{ic(g_i)}^{(u,r_u)}$
$(t,1)$	$\lvert C_{H_t}(g_{it1})\rvert$	$a_{i1}^{(t,1)}$	$a_{i2}^{(t,1)}$	$\cdots$	$a_{ic(g_i)}^{(t,1)}$
$(t,2)$	$\lvert C_{H_t}(g_{it2})\rvert$	$a_{i1}^{(t,2)}$	$a_{i2}^{(t,2)}$	$\cdots$	$a_{ic(g_i)}^{(t,2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
(t,r_t)	$\lvert C_{H_t}(g_{itr_{it}})\rvert$	$a_{i1}^{(t,r_t)}$	$a_{i2}^{(t,r_t)}$	$\cdots$	$a_{ic(g_i)}^{(t,r_t)}$
m_{ij}		m_{i1}	m_{i2}	$\cdots$	$m_{ic(g_i)}$

Properties of Fischer Matrices

The Fischer matrices satisfy some interesting properties, which help in computations of their entries.

(i) $\sum_{k=1}^{t} c(g_{ik}) = c(g_i)$,

(ii) $\mathcal{F}_i$ is non-singular for each i,

(iii) $a_{ij}^{(1,1)} = 1, \ \forall \ 1 \leq j \leq c(g_i)$,

(iv) $a_{11}^{(k,m)} = [G : H_k]\theta_k(1_N), \ \forall \ (k,m) \in J_1$,

(v) For each $1 \leq i \leq r$, the weights m_{ij} satisfy the relation $\sum_{j=1}^{c(g_i)} m_{ij} = |N|$,

Properties of Fischer Matrices

(vi) **Column Orthogonality Relation:**

$$\sum_{(k,m)\in J_i} |C_{H_k}(g_{ikm})| a_{ij}^{(k,m)} \overline{a_{ij'}^{(k,m)}} = \delta_{jj'} |C_{\overline{G}}(g_{ij})|,$$

(vii) **Row Orthogonality Relation:**

$$\sum_{j=1}^{c(g_i)} m_{ij} a_{ij}^{(k,m)} \overline{a_{ij}^{(k',m')}} = \delta_{(k,m)(k',m')} a_{i1}^{(k,m)} |N|.$$

Example of the Fischer Matrices

- Corresponding to $[2_2]_{(3^{1+2}:8):2}$, the Fischer matrix of $\overline{G} = 2^{1+6}_{-}:((3^{1+2}:8):2)$ will have the form:

		$\mathcal{F}_3$				
$g_3 = {}^{2_2}$		g_{31}	g_{32}	g_{33}	g_{34}	g_{35}
$o(g_{3j})$		8	8	2	4	8
$\lvert C_{\overline{G}}(g_{3j})\rvert$		192	192	96	32	32
(k,m)	$\lvert C_{H_k}(g_{3km})\rvert$					
$(1,1)$	12	1	1	1	1	1
$(2,1)$	12	$2\sqrt{2}i$	$-2\sqrt{2}i$	0	0	0
$(3,1)$	4	3	3	3	-1	-1
$(4,1)$	12	1	1	-1	-1	1
$(4,2)$	4	3	3	-3	1	-1
m_{3j}		8	8	16	48	48

- By [8] the identity Fischer matrix $\mathcal{F}_1$ of the non-split extension group $\overline{G}_n = 2^{2n}\cdot Sp(2n,2)$ for any $n \in \mathbb{N}^{\geq 2}$ will have the form:

		$\mathcal{F}_1$	
$g_1 = {}^{1_{Sp(2n,2)}}$		g_{11}	g_{12}
$o(g_{1j})$		1	2
$\lvert C_{\overline{G}}(g_{1j})\rvert$		$\lvert\overline{G}_n\rvert$	$\lvert\overline{G}_n\rvert/2^{2n}-1$
(k,m)	$\lvert C_{H_k}(g_{1km})\rvert$		
$(1,1)$	$\lvert G_n\rvert$	1	1
$(2,1)$	$\lvert G_n\rvert/2^{2n}-1$	$2^{2n}-1$	-1
m_{1j}		1	$2^{2n}-1$

Some Survey On Clifford-Fischer Theory

- Professor J. Moori has a significant contribution to this domain. Indeed he developed the coset analysis technique in his PhD thesis [15] and in [16].
- Then together with his MSc and PhD students, they enriched this area of research by applying the coset analysis and Clifford-Fischer theory to many various split and non-split group extensions in a considerable number of publications. For example, but not limited to, one can refer to [1], [4, 5, 6, 7, 8, 9, 10], [17, 18], [20, 21, 22, 23], [25] or [26].
- Barraclough produced an interesting PhD thesis [2], which contained a chapter on the method of Clifford-Fischer theory. He used this method to find the character table of any group of the form $2^{2\cdot}G{:}2$ for any finite group G.
- Also in 2007, H. Pahlings [24] calculated the Fischer matrices and the character table of the non-split extension $2_+^{1+22}\cdot Co_2$, which is the second largest maximal subgroup of the Baby Monster group $\mathbb{B}$.
- Then in 2010, H. Pahlings together with his student K. Lux published an interesting book [14] containing a full chapter on Clifford-Fischer theory that includes several examples on the application of the method.

The Bibliography

1. F. Ali and J. Moori, The Fischer Clifford matrices of a maximal subgroup of Fi'_{24}, *Journal of Representation Theory* **7** (2003), 300 - 321.
2. R. W. Barraclough, *Some Calculations Related To The Monster Group* (PhD Thesis, University of Birmingham, Birmingham 2005).
3. A. B. M. Basheer, *Clifford-Fischer Theory Applied to Certain Groups Associated with Symplectic, Unitary and Thompson Groups* (PhD Thesis, University of KwaZulu-Natal, Pietermaitzburg 2012). Accessed online through "http://researchspace.ukzn.ac.za/xmlui/handle/10413/6674?show=full".
4. A. B. M. Basheer and J. Moori, Fischer matrices of Dempwolff group $2^5 \cdot GL(5,2)$, *International Journal of Group Theory* **1** No. 4 (2012), 43 - 63.
5. A. B. M. Basheer and J. Moori, On the non-split extension group $2^6 \cdot Sp(6,2)$, *Bulletin of the Iranian Mathematical Society*, to appear.
6. A. B. M. Basheer and J. Moori, Fischer matrices of the group $2^{1+8}_{+} \cdot A_9$, submitted.
7. A. B. M. Basheer and J. Moori, On a group of the form $3^7{:}Sp(6,2)$, submitted.
8. A. B. M. Basheer and J. Moori, On the non-split extension $2^{2n} \cdot Sp(2n,2)$ and the character table of $2^8 \cdot Sp(8,2)$, to be submitted.
9. A. B. M. Basheer and J. Moori, On a group of the form $2^{10}{:}(U_5(2){:}2)$, to be submitted.
10. A. B. M. Basheer and J. Moori, Clifford-Fischer theory applied to a group of the form $2^{1+6}_{-}{:}((3^{1+2}{:}8){:}2)$, to be submitted.
11. B. Fischer, *Clifford matrizen, manuscript* (1982).
12. B. Fischer, *Unpublished manuscript* (1985).
13. B. Fischer, Clifford matrices, *Representation theory of finite groups and finite-dimensional Lie algebras* (eds G. O. Michler and C. M. Ringel; Birkhäuser, Basel, 1991), 1 - 16.

The Bibliography

14. K. Lux and H. Pahlings, *Representations of Groups: A Computational Approach* (Cambridge University Press, Cambridge 2010).
15 J. Moori, *On the Groups G^+ and $\overline{G}$ of the form $2^{10}{:}M_{22}$ and $2^{10}{:}\overline{M}_{22}$* (PhD Thesis, University of Birmingham 1975).
16 J. Moori, On certain groups associated with the smallest Fischer group, *J. London Math. Soc.* **2** (1981), 61 - 67.
17. J. Moori and Z. Mpono, The Fischer-Clifford matrices of the group $2^6{:}SP_6(2)$, *Quaest. Math.* **22** (1999), no. 2, 257 - 298.
18. J. Moori, Z. E. Mpono and T. T. Seretlo, A group $2^7{:}S_8$ in $\overline{Fi}_{22}$, *South East Asian Bulletin of Mathmatics*, to appear.
19. J. Moori and T. T. Seretlo, On the Fischer-Clifford matrices of a maximal subgroup of the Lyons Group Ly, *Bulletin of the Iranian Maths Soc.*, to appear.
20. J. Moori and T. T. Seretlo, A Group of the form $2^6{:}A_8$ as an inertia factor group of $2^8{:}O_8^+(2)$, submitted.
21. J. Moori and T. T. Seretlo, On two non-split extension groups associated with HS and $\mathrm{HS}{:}2$, submitted.
22. J. Moori and T. T. Seretlo, On an inertia factor group of $O_{10}^+(2)$, submitted.
23. J. Moori and K. Zimba, Fischer-Clifford Matrices of $B(2,n)$, *Quaestiones Math.* **29** (2005), 9 - 37.
24. H. Pahlings, The character table of $2_+^{1+22}{\cdot}Co_2$, *Journal of Algebra* **315** (2007), 301 - 325.
25. B. G. Rodrigues, *On The Theory and Examples of Group Extensions* (MSc Thesis, University of Natal, Pietermaritzburg 1999).
26. N. S. Whitely, *Fischer Matrices and Character Tables of Group Extensions* (MSc Thesis, University of Natal, Pietermaritzburg 1993).

Acknowledgment

I would like to thank my Postgraduate Diploma, Master, PhD and Postdoctoral supervisor Professor **Jamshid Moori**.

Thank you!