A generalisation on the solvability of finite groups with three class sizes for normal subgroups

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(in collaboration with María José Felipe)

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New Topic

Influence of $\mathrm{cs}_G(N)$ on the structure of N

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Example

Let $G = S_3 \wr \mathbb{Z}_2$ and let $N = S_3 \times S_3 \trianglelefteq G$. Then

$$\mathrm{cs}(N) = \{1, 2, 3, 4, 6, 9\}, \text{ while } \mathrm{cs}_G(N) = \{1, 4, 6, 9, 12\}.$$

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Theorem (Itô, 1953)

If $\mathrm{cs}(G) = \{1, m\}$ then $m = p^a$ for some prime p and $G = P \times A$, with A abelian, and P a p-group.

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Theorem (Alemany, Beltrán, Felipe)

Suppose that N is a normal subgroup of a group G having two G-class sizes, then either N is abelian or $N = P \times A$, with P a p-group and $A \subseteq \mathbf{Z}(G)$.

E. Alemany, A. Beltrán, M.J. Felipe, Nilpotency of normal subgroups having two G-class sizes. Proc. Amer. Math. Soc., 139 (2011), 2663-2669.

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Definition. A nonabelian group G is said to be an F-group if for every $x, y \in G \setminus \mathbf{Z}(G)$, such that $\mathbf{C}_G(x) \subseteq \mathbf{C}_G(y)$, then $\mathbf{C}_G(x) = \mathbf{C}_G(y)$.

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• A. Camina (1974) shows that any group with three class sizes which is not an F-group is a direct product of an abelian group and a group whose order involves no more than two primes.

Theorem (Dolfi, Jabara, 2009)

A finite group G has three class sizes if and only if, up to an abelian factor, either

(1) G is a p-group for some prime p or

(2) $G = KL$ with $K \trianglelefteq G$, $(|K|, |L|) = 1$ and one of the following occurs

   (a) both K and L are abelian, $\mathbf{Z}(G) < L$ and G is a quasi-Frobenius group,

   (b) K is abelian, L is a non-abelian p-group, for some prime p and $\mathbf{O}_p(G)$ is an abelian subgroup of index p in L and $G/\mathbf{O}_p(G)$ is a Frobenius group or

   (c) K is a p-group with two class sizes for some prime p, L is abelian, $\mathbf{Z}(K) = \mathbf{Z}(G) \cap K$ and G is quasi-Frobenius.

Solvability of $N \trianglelefteq G$ with $\mathrm{cs}_G(N) = \{1, m, n\}$

The case in which m does not divide n

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Definition

A non-central normal subgroup N of a group G is said to be an F-**normal subgroup** if for every $x, y \in N \setminus \mathbf{Z}(G)$, such that $\mathbf{C}_G(x) \subseteq \mathbf{C}_G(y)$, then $\mathbf{C}_G(x) = \mathbf{C}_G(y)$.

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Lemma

If N is an F-normal subgroup of a group G, then $N/(N \cap \mathbf{Z}(G))$ has a non-trivial normal abelian partition.

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We use results of Baer and Suzuki on groups having a non-trivial normal partition to clasify F-normal subgroups.

The case in which m does not divide n

Theorem (Akhlaghi, Beltrán, Felipe, Khatami)

Let G be a group and N be an F-normal subgroup of G. Then N satisfies one of the following conditions:

(1) $N/\mathbf{Z}(N)$ is a Frobenius group, with Frobenius kernel $L/\mathbf{Z}(N)$ and complement $K/\mathbf{Z}(N)$, with K and L abelian.

(2) $N/\mathbf{Z}(N)$ is a Frobenius group, with kernel $L/\mathbf{Z}(N)$ and complement $K/\mathbf{Z}(N)$, where K is abelian, and $L/\mathbf{Z}(N)$ is of prime-power order, and L is an F-normal subgroup.

(3) $N/\mathbf{Z}(N) \cong S_4$ and V is non-abelian, for $V/\mathbf{Z}(N)$, the Klein four-group of $N/\mathbf{Z}(N)$. In particular, N is an F-group.

(4) N has abelian Fitting subgroup of index p, p divides $|\mathbf{F}(N)/\mathbf{Z}(N)|$, and N is an F-group.

(5) $N = P \times \mathbf{Z}(N)_{p'}$, where $P \in \mathrm{Syl}_p(N)$.

(6) $N/\mathbf{Z}(N) \cong \mathrm{PSL}(2, p^h)$ or $\mathrm{PGL}(2, p^h)$, where $p^h \geq 4$.

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Theorem (Akhlaghi, Beltrán, Felipe, Khatami)

Let N be an F-normal subgroup of G such that $|\mathrm{cs}_G(N)| = 3$. Then N is solvable. In particular, when $\mathrm{cs}_G(N) = \{1, m, n\}$ and m does not divide n, then N is solvable.

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The proof consists in showing that case (6) in the above classification cannot happen.

Z. Akhlaghi, A. Beltrán, M.J. Felipe, M. Khatami, Structure of normal subgroups with three G-class sizes. Monatsh. Math. **167** (2012), no 1, 1-12.

Theorem (Akhlaghi, Beltrán, Felipe, Khatami)

Let N be a normal subgroup of a finite group G such that $\mathrm{cs}_G(N) = \{1, m, n\}$, where $m < n$ and m does not divide n. Then one of the following conditions is satisfied:

(1) $N = P \times A$, where $P \in \mathrm{Syl}_p(N)$, p prime and $A \subseteq \mathbf{Z}(G)$.

(2) $N/\mathbf{Z}(N)$ is a Frobenius group, with Frobenius kernel $L/\mathbf{Z}(N)$ and Frobenius complement $K/\mathbf{Z}(N)$, and

  (a) either K and L are abelian, and

$$\mathrm{cs}(N) = \{1, |L/\mathbf{Z}(N)|, |K/\mathbf{Z}(N)|\}.$$

  (b) or K is abelian, and $L/\mathbf{Z}(N)$ is of prime-power order, and

$$\mathrm{cs}(N) = \{1, |L/\mathbf{Z}(N)|,\ |K/\mathbf{Z}(N)||x^L|\ : x \in L \setminus \mathbf{Z}(N)\}.$$

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Thus, $\mathbf{C}_N(z) \trianglelefteq \mathbf{C}_G(z)$ and this normal subgroup has at most two p-regular $\mathbf{C}_G(z)$-class sizes.

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Thus, $\mathbf{C}_N(z) \trianglelefteq \mathbf{C}_G(z)$ and this normal subgroup has at most two p-regular $\mathbf{C}_G(z)$-class sizes.

Theorem (Akhlaghi, Beltrán, Felipe, J. Group Theory, 2013)

Let $N \trianglelefteq G$ having exactly two G-class sizes of p-regular elements. Then N is solvable. Moreover, either N has abelian p-complements or all p-regular elements of $N/(N \cap \mathbf{Z}(G))$ have prime power order.

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Theorem

Suppose that N is a solvable normal subgroup of a group G and suppose that an integer m divides s for every $s \in \mathrm{cs}_G(N)$, $s \neq 1$. If $g \in N$ and $|g^G| = m$, then $g \in \mathbf{F}(N)$.

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Theorem

Suppose that N is a nonsolvable normal subgroup of a group G and suppose that an integer m divides $|x^G|$ for every $x \in N \setminus \mathbf{Z}(N)$. Then m divides $|\mathbf{Z}(N)|$.

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If N is a nonabelian normal subgroup of a finite group G and $|\mathrm{cs}_G(N)| = 3$, then $\mathbf{Z}(N)$ is properly contained in $\mathbf{F}(N)$.

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A. Beltrán, M.J. Felipe, Solvability of normal subgroups and G-class sizes. Publ. Math. Deb. In print.

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Theorem (Isaacs, 1970)

Let N be a normal subgroup of a group G such that all of the conjugacy classes of G which lie outside N have equal sizes. Then G/N is cyclic or else every nonidentity element of G/N has prime order.

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Definition

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Theorem (Akhlaghi, Beltrán, Felipe)

Let N/K be a normal section satisfying $(^*)$ over G.

i) If $\mathbf{Z}(N) \nsubseteq K$, then N/K is a p-group for some prime p and N/K is either abelian or has exponent p.

ii) If $\mathbf{Z}(N) \subseteq K$, then either N/K is cyclic or is a CP-group. In the first case, N has abelian Hall π-subgroups and normal π-complement, where $\pi = \pi(N/K)$.

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Theorem (Heineken, 2006)

If G is a finite non-solvable CP-group, then there exist normal subgroups B, C of G such that $1 \subseteq B \subseteq C \subseteq G$ and B is a 2-group, C/B is non-abelian and simple, and G/C is a p-group for some prime p and cyclic or generalised quaternion. In particular, if G is a finite non-abelian simple CP-group, then G is isomorphic to: $\mathrm{L}_2(q)$, for $q = 5, 7, 8, 9, 17$, $\mathrm{L}_3(4)$, $\mathrm{Sz}(8)$ or $\mathrm{Sz}(32)$.

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• By induction on $|N|$ we have that N is perfect. If $N' < N$, then $|\mathrm{cs}_G(N')| \leq 3$, so N' is solvable and N is solvable as well.

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• Therefore, there exists $B \trianglelefteq N$, such that N/B is simple (CP-group) and $B/\mathbf{F}(N)$ is a 2-group
• We make a case-by-case analysis for each of the simple groups, and we finally get a contradiction.

Open problem

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When dealing with G-class sizes and normal subgroups, such structure does not hold.

Thank you for your attention