Shift Dynamics and Asphericity for Cyclically Presented Groups

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Ask a simple question...

J. H. Conway, Advanced Problem 5327, Amer. Math. Monthly **72** (1965), 915.

Show that the group generated by a, b, c, d, e *subject to* $ab = c$, $bc = d$, $cd = e$, $de = a$, $ea = b$ *is cyclic of order* 11.

This group is Fib_5.

Many interesting responses!

In 1967 the Monthly reported solutions from

- correspondents on five continents (**plus Scotland**),
- in 52 separate communications,
- and including solutions from Ahmad, Baron, Brisley, Bumby, **Campbell**, Conway, Coxeter, Flanders, Imrich, Lyndon, Mendelsohn, Moser, Sinkov, Wicks....

Cyclic Symmetry

$$\mathrm{Fib}_5 = \langle a, b, c, d, e \;: \\ ab = c, bc = d, cd = e, de = a, ea = b\rangle$$

has a **shift automorphism** of exponent 5:

$$a \mapsto b \mapsto c \mapsto d \mapsto e \mapsto a$$

Lyndon's Argument that $\mathrm{Fib}_5 \cong C_{11}$

- $\mathrm{Fib}_5 = \langle a, b \rangle$ (easy),
- $a^5 = b^4$ is central (clever manipulations),
- $b^5 = c^4$ is central (**shift!!!**),
- so b is central,
- so Fib_5 is abelian.

A Rich Story Unfolds

$\mathrm{Fib}_n = \langle x_i \ : \ x_i x_{i+1} x_{i+2}^{-1} \rangle$ is:

- Finite iff $n = 1, 2, 3, 4, 5,$ or 7 (1967-1990);
- Infinite if $n \geq 11$ [Lyndon 1974];
- Infinite if $n = 8$ or 10 [Brunner 1974];
- Infinite and automatic if $n = 9$ [Havas-Richardson-Sterling 1979, Newman 1990, Holt 1995];

And More

$\mathrm{Fib}_n = \langle x_i \ : \ x_i x_{i+1} x_{i+2}^{-1} \rangle$ is:

- The fundamental group of a closed orientable hyperbolic three-manifold if $n \geq 8$ is even [Helling-Kim-Mennicke 1994];
- NOT the fundamental group of a hyperbolic three-orbifold of finite volume if n is odd [Maclachlan 1995].

Cyclically Presented Groups and the Shift

$$\begin{aligned} w \in F &= \mathrm{Free}(x_0, \ldots, x_{n-1}) \\ \theta(x_i) &= x_{i+1} \\ \mathcal{P}_n(w) &= \langle x_i \; : \theta^i(w) \rangle \; (i \mod n) \\ G_n(w) &= F/\langle\langle \theta^i(w) \rangle\rangle \\ \text{Shift} &: \; \theta^n = 1 \text{ in } \mathrm{Aut}(G_n(w)) \end{aligned}$$

Example. $\mathrm{Fib}_n = G_n(x_0 x_1 x_2^{-1})$

Theorem. [D. Johnson, 1974] *For all n, the shift θ has order n in* $\mathrm{Aut}(\mathrm{Fib}_n)$. *(I.e. C_n acts faithfully via the shift on* Fib_n.*)*

Main Results

Theorem A. *If $\mathcal{P}_n(w)$ is orientable and combinatorially aspherical, then C_n acts freely via the shift on the nonidentity elements of $G_n(w)$.*

Theorem B(xxx). *$\mathcal{P}_n(x_0x_kx_l)$ is combinatorially aspherical if and only if C_n acts freely via the shift on the nonidentity elements of $G_n(x_0x_kx_l)$.*

Theorem C(xxx). *$G_n(x_0x_kx_l)$ is finite if and only if the shift θ has a nonidentity fixed point.*

Theorems B and C rely on [Edjvet-Williams 2010] and [B-Pride 1990].

Orientability is Necessary

Definition. *$\mathcal{P}_n(w)$ is **orientable** if w is not a cyclic permutation of the inverse of any of its shifts.*

Lemma. *$\mathcal{P}_n(w)$ is not orientable if and only if $n = 2m$ is even and w is freely equal to a word of the form $u\theta^m(u)^{-1}$.*

Note that θ^m fixes $u \in G_{2m}(u\theta^m(u)^{-1})$.

Example. The shift acts as the identity on $G_2(x_0x_1^{-1}) \cong C_\infty$.

Identities Among Relations

Given $\mathcal{P} = \langle \mathbf{x} \ : \ \mathbf{r} \rangle$ for G, let $F = \mathrm{Free}(\mathbf{x})$ and define $\mathrm{Free}(F \times \mathbf{r}) \xrightarrow{\partial} F$ by

$$\partial(w, r) = wrw^{-1}.$$

The group $\mathbb{I} = \ker \partial$ of **identity sequences** contains the normal closure $\mathbb{P}$ of **Peiffer identities**:

$$(w, r)^{\epsilon}(v, s)^{\delta}(w, r)^{-\epsilon}(wrw^{-1}v, s)^{-\delta}.$$

Then $\mathbb{I}/\mathbb{P} \cong \pi_2 K$ as $\mathbb{Z}G$-modules where $K = \bigvee_{\mathbf{x}} S^1_x \cup \bigcup_{\mathbf{r}} c^2_r$ is the cellular model of $\mathcal{P}$. [Reidemeister 1949]

Definition. *$\mathcal{P}$ is **combinatorially aspherical** (CA) if $\mathbb{I}/\mathbb{P}$ is $\mathbb{Z}G$-generated by (classes of) length two identity sequences.*

Examples. $\mathcal{P}$ is CA if $\pi_2 K \cong \mathbb{I}/\mathbb{P} = 0$ (topological asphericity). One-relator presentations $\langle \mathbf{x} : \dot{W}^e \rangle$ are CA (Lyndon 1950).

$$(1, \dot{W}^e)(\dot{W}, \dot{W}^e)^{-1}$$

Also, $\mathcal{P}_3(x_0 x_1 x_2)$ is a CA presentation for the free group $F = F(x_0, x_1)$. As in Thm A, the order three automorphism of F with

$$x_0 \mapsto x_1 \text{ and } x_1 \mapsto (x_0 x_1)^{-1}$$

has no nontrivial fixed points.

A Look Inside Theorem A

The shift is realized as conjugation by a in $E = G_n(w) \rtimes_\theta C_n = \langle a, x \;:\; a^n, W(a,x)\rangle$ where $x_i = a^i x a^{-i}$. There is a C_n-covering

$$p : \widehat{M} \to M = K(C_n, 1) \vee S^1_x \cup c^2_{W(a,x)}$$

where $p^{-1}(K(C_n,1)) \simeq *$. Collapsing this pre-image to a point yields the cellular model K of the cyclic presentation $\mathcal{P}_n(w)$.

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\simeq} & K \\ p \downarrow & & \\ M & & \end{array}$$

Thus $\pi_2 M \cong \pi_2 K \cong \mathbb{I}/\mathbb{P}$.

Special Case: $W(a,x)$ is not a proper power

Then orientable $\mathcal{P}_n(w)$ is CA iff $M = K(E,1)$ for $E = G_n(w) \rtimes_\theta C_n$. (When $W(a,x)$ is a proper power, add a cell in each dimension three and up to obtain a $K(E,1)$.)

Now $H^k(E;-) \cong H^k(C_n;-)$ $(k \geq 3)$. By a Theorem of Serre (1979),

- $G_n(w)$ is torsion-free and
- $gC_ng^{-1} \cap C_n \neq \{1\} \Rightarrow g \in C_n$.

So no nontrivial power of $a \in E$ centralizes any nontrivial element of $G_n(w)$. QED

Moral of Theorem A

If any power of the shift has a nontrivial fixed point, then $\mathcal{P}_n(w)$ supports an "interesting" identity sequence (i.e. spherical map).

$$\begin{aligned} E &= \langle a, x \ : a^n,\ W(a,x) \rangle \\ &= \langle a, x \ : a^n,\ x^{\epsilon_1} a^{p_1} \ldots x^{\epsilon_L} a^{p_L} \rangle \end{aligned}$$

Theorem. *If* $n \mid f \sum \epsilon_i + \sum p_i$ *then*

$$\nu^f(a) = a \text{ and } \nu^f(x) = a^f$$

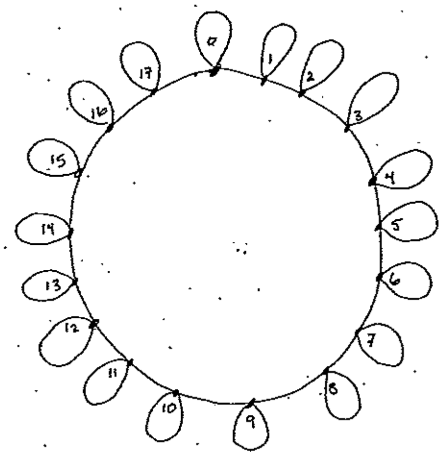
defines a retraction $\nu^f : E \to C_n = \langle a \rangle$. *The kernel* G^f *of any such retraction is cyclically presented and the* C_n*-action on* G^f *via the shift is isomorphic to the* C_n *action on* E/C_n.

Moral. Cyclically presented kernels corresponding to different retractions are commensurable (in E) and have identical shift dynamics.

Example: $E = \langle a, x \ : \ a^{18}, xaxa^5xa^{-6} \rangle$

Retractions $\nu^0, \nu^6, \nu^{12} : E \to C_{18}$ determine C_{18}-covers of M.

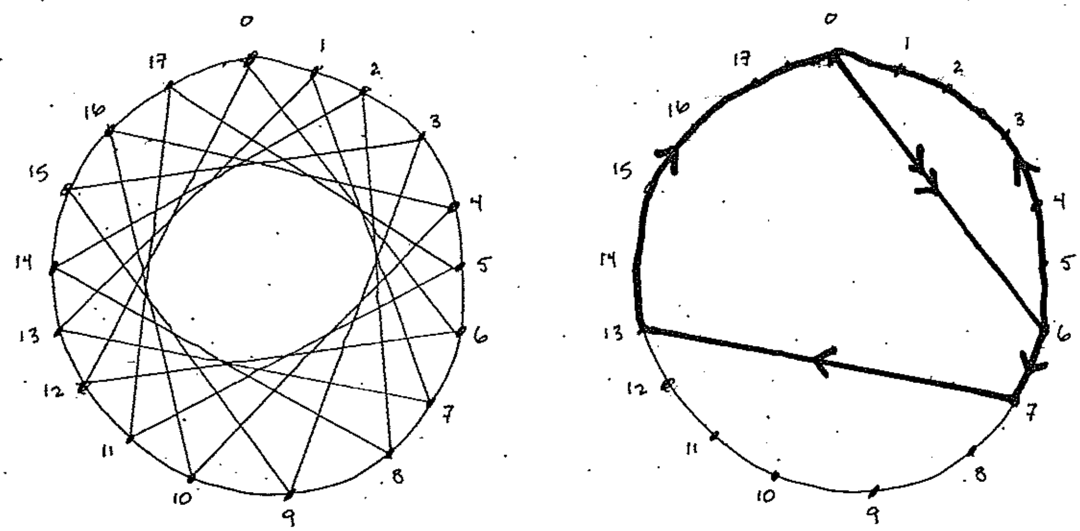
$\widehat{M}^0$ $\quad\quad$ $\ker \nu^0 = G^0 \cong G_{18}(x_0 x_1 x_6)$



G^0, G^6, G^{12} all have index 18 in E; all are cyclically presented with identical shift dynamics.

Example: $E = \langle a, x \ : \ a^{18}, xaxa^5xa^{-6} \rangle$ **(cont'd)**

$\widehat{M}^6$



$G^6 \cong G_{18}(x_0^2 x_7) \cong C_s$ where $s = 2^{18} - 1$.

$$x_0^2 x_7 = x_7^2 x_{14} = \cdots \ x_4^2 x_{11} = x_{11}^2 x_0 = 1$$

$$\theta^7(x_0) \quad = \quad x_0^{-2}$$

$$\theta^7(x_0^{s/3}) \quad = \quad x_0^{s/3}$$

Metacyclic $G_n(x_0x_kx_l)$

Theorem. The cyclically presented group $G_{3m}(x_0x_1x_m)$

- is metacyclic of order $2^{3m}-(-1)^m$ [Edjvet-Williams 2010] and
- has a fixed point subgroup of order three under the shift.

A Group of Order $342 = 2 \cdot 3^2 \cdot 19$

$$\begin{aligned} E &= \langle a, x \ : \ a^{18}, xaxa^4xa^{-5} \rangle \\ &\overset{u=a^5x^{-1}}{=} \langle a, u \ : \ a^{18}, u^2a^9ua^6 \rangle \\ &= J *_{b=a^3} C_{18} \text{ where} \end{aligned}$$

$J = \langle b, u \ : \ b^6, u^2b^3ub^2 \rangle = C_{19} \rtimes_{-4} C_{18}$ is a Frobenius group of order $|J| = 342$ [GAP,B-Pride 1990]. From a suitable retraction:

$$E = G_{18}(x_0x_1x_5) \rtimes_\theta C_{18}.$$

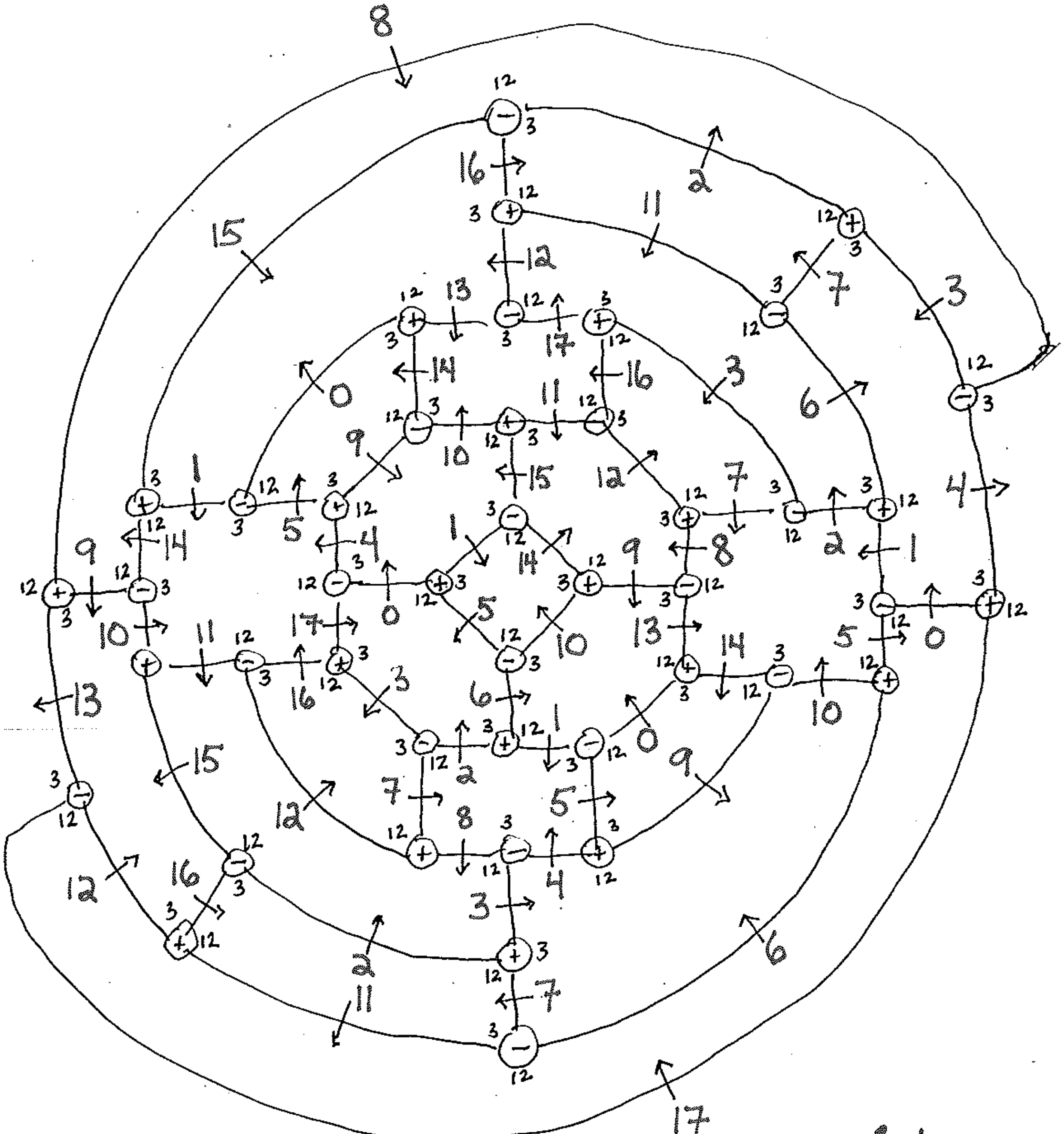
$E = G_{18}(x_0x_1x_5) \rtimes_\theta C_{18} = J *_{b=a^3} C_{18}$

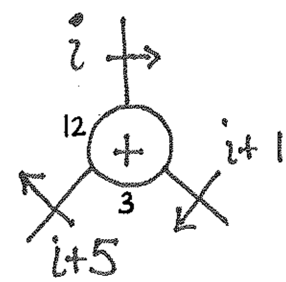
The shift on $G_{18}(x_0x_1x_5)$ is fixed point free (viz Thm C).

$|J/C_6| = 19 \cdot 3$, so C_6 does not act freely on $J/C_6 - \{1C_6\}$.

The shift action by C_{18} is faithful but not free on the infinite group $G_{18}(x_0x_1x_5)$.

Thm B demands an interesting identity sequence (or sphere).





$E = (C_{18}, x : \; x^2 a^3 x a^{12})$

$G_{18}(1,5)$

A Class of Cyclically Presented Groups

$$\begin{aligned} J_m &= \langle b, u \ : \ b^6, u^{m-1}b^3ub^2 \rangle \\ E_m &= J_m *_{b=a^m} C_{6m} \\ &= G_{6m}(x_0x_1 \ldots x_{m-1}x_{4m-1}) \rtimes_\theta C_{6m} \end{aligned}$$

m	$\|J_m\|$
2	12
3	342
4	4632
5	38010
6	235908
7	?

The Story Continues?

The shift on the cyclically presented groups

$$G_{6m}(x_0 x_1 \dots x_{m-1} x_{4m-1})$$

is fixed point free for all m. For $3 \leq m \leq 6$, they have nontrivial torsion and so are not CA; the corresponding cyclic presentations support interesting, yet unseen, identity sequences.