Approximate groups and their applications: part 2

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St. Andrews, August 3-10, 2013

The sum-product phenomenon

A precursor (historically) to approximate groups is the following result:

Theorem (Bourgain-Katz-Tao, 2003)

$\forall \delta > 0$, $\exists \varepsilon > 0$ *s.t. if* A *is an arbitrary subset of the finite field* $\mathbb{F}_p$ *(*p *any prime), then*

$$|AA| + |A + A| > |A|^{1+\varepsilon}$$

unless $|A| > p^{1-\delta}$.

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A similar result says that $\exists \varepsilon > 0$ s.t. for every subset $A \subset \mathbb{F}_p$,

$$|A^2 + A^2 + A^2| \geqslant \min\{|\mathbb{F}_p|, |A|^{1+\varepsilon}\}$$

The proof of the sum-product theorem goes by finding a large subset $A' \subset A$ which does not grow much under all operations (addition, multiplication, division), i.e.

$$\left|\frac{A'^{\pm k} \pm \ldots \pm A'^{\pm k}}{A'^{\pm k} \pm \ldots \pm A'^{\pm k}}\right| \ll |A'|^{1+\varepsilon}$$

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There are variants of the sum-product theorem, where one considers $|AA + A|$ instead of $|AA| + |A + A|$ or other similar expressions.

One can also define a notion of K-approximate field, and show that they are either bounded in size or form a significant proportion of genuine finite field.

The sum-product phenomenon and approximate subgroups of the affine group

It turns out (an observation of Helfgott) that one can see the sum-product phenomenon as a special case of the classification of approximate subgroups of the affine group $G_p := \{ax + b\}$ over $\mathbb{F}_p$.

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Indeed set

$$B := \begin{pmatrix} A & A \\ 0 & 1 \end{pmatrix} \subset G_p = \begin{pmatrix} \mathbb{F}_p^\times & \mathbb{F}_p \\ 0 & 1 \end{pmatrix}$$

Then, if $|AA + A| \leqslant K|A|$, (later K will be $K = |A|^\varepsilon$) then

$$|BB| \leqslant K^2|B|.$$

So B has doubling at most K^2, hence is roughly equivalent to a CK^C-approximate subgroup of the affine group $\{ax + b\}$.

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But B is not of this type if $K = |A|^\varepsilon$ for small enough $\varepsilon > 0$. So we must have

$$|AA + A| > |A|^{1+\varepsilon}$$

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Approximate subgroups of linear groups

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In 2005, motivated by the new method of Bourgain-Gamburd for expanders, H. Helfgott proved the following:

Theorem (H. Helfgott's product theorem, 2005)

$\forall \delta > 0$, $\exists \varepsilon > 0$, s.t. if $A \subset \mathrm{SL}_2(\mathbb{F}_p)$ (p any prime) be any generating subset, then

$$|AAA| > |A|^{1+\varepsilon}$$

unless $|A| > |\mathrm{SL}_2(\mathbb{F}_p)|^{1-\delta}$.

Remark: why AAA and not AA ? here is a counter-example: take $A = H \cup \{x\}$, where

$$H := \begin{pmatrix} 1 & \mathbb{F}_p \\ 0 & 1 \end{pmatrix}, \text{ and } x := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

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then $AA = H \cup xH \cup Hx \cup \{x^2\}$, while $xHx^{-1} \cap H = \{1\}$, and thus

$$|AA| = 3|H| + 1 = 3|A| - 2,$$

while

$$|AAA| \geqslant |HxH| = |H|^2 = (|A| - 1)^2.$$

Approximate subgroups of linear groups

Translation of Helfgott theorem in terms of approximate groups:

Theorem (Helfgott reformulated)

Let $A \subset \mathrm{SL}_2(\mathbb{F}_p)$ be a K-approximate subgroup which generates $\mathrm{SL}_2(\mathbb{F}_p)$. Then either $|A| < CK^C$, or $|A| > |\mathrm{SL}_2(\mathbb{F}_p)|/CK^C$.

Here $C > 0$ is an absolute constant (independent of p).

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how to get Helfgott's theorem from this: if $|AAA| < |A|^{1+\varepsilon}$, then set $K = |A|^{\varepsilon}$. Now A will be CK^C-roughly equivalent to an CK^C-approximate group $B \supset A$. If $|B| < CK^C$, then we must have $|A| \leqslant CK^C|B| \leqslant C^2K^{2C} < C^2|A|^{2C\varepsilon}$, which implies that $|A|$ is bounded if $2C\varepsilon < 1$. If on the other hand $|B| > |\mathrm{SL}_2(\mathbb{F}_p)|/CK^C$, then $|A| > |B|/CK^C > |\mathrm{SL}_2(\mathbb{F}_p)|/C^2K^{2C}$, so $|A| > |\mathrm{SL}_2(\mathbb{F}_p)|^{\frac{1}{1-2C\varepsilon}}$. Done.

Here is another way to reformulate yet again Helfgott's theorem:

Theorem (Helfgott reformulated one more time)

Every generating K-approximate subgroup of $\mathrm{SL}_2(\mathbb{F}_p)$ is CK^C-roughly equivalent to either $\{1\}$ or $\mathrm{SL}_2(\mathbb{F}_p)$.

In other words: There are no non trivial generating approximate subgroups of $\mathrm{SL}_2(\mathbb{F}_p)$.

Helfgott's proof

Helfgott's proof is based on the sum-product phenomenon:

- first show that $|tr(A)| \gg |A|^{\frac{1}{3}}$,
- show that there is a set $V \subset A^{O(1)}$ of simultaneously diagonalisable elements s.t. $|V| \simeq |tr(A)|$,
- (trace amplification) show that, for some $a \in A^{O(1)}$, $|tr(VaVa^{-1})| \gg |V|^{1+\varepsilon}$,
- conclude using step 2 again and showing that $|VbVb^{-1}V| \gg |V|^3$ for some $b \in A^{O(1)}$.

Pyber-Szabo and (simultaneously) B-Green-Tao proved the following extension of Helfgott's result:

Theorem (Product theorem for finite simple groups of Lie type)

$\forall \delta > 0$, $\exists \varepsilon = \varepsilon(\delta, r) > 0$ *such that if A is any generating subset of a finite simple (or quasi-simple) group of Lie type $\mathbf{G}(q)$ with rank at most r, one has:*

$$|AAA| > |A|^{1+\varepsilon}$$

unless $|A| > |\mathbf{G}(q)|^{1-\delta}$.

Generalization to all finite fields and all Lie type

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In fact, if $\mathbf{G}(q)$ is simple, one can show (Gowers' trick) that $AAA = \mathbf{G}(q)$ if $|A| > |\mathbf{G}(q)|^{1-\delta}$ for $\delta = \delta(r) > 0$ small enough.

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(Pyber) The analogous statement fails for the symmetric (or alternating) groups: take inside $G = S_n$

$$A = H \cup \{\sigma^{\pm 2}\},$$

where $\sigma = (1, ..., n)$ a long cycle (say n odd), and
$H := \langle (1,2) \rangle \cdot \ldots \cdot \langle ([\frac{n}{2}], [\frac{n}{2}] + 1) \rangle \simeq (\mathbb{Z}/2\mathbb{Z})^{[\frac{n}{2}]}$.
Then A is a generating 4-approximate subgroup.

Reformulation in terms of approximate groups

As before, the above product theorem can be reformulated in terms of approximate subgroups:

Theorem (classification of approximate subgroups of $\mathbf{G}(q)$)

If A is a generating K-approximate subgroup of $\mathbf{G}(q)$, then either $|A| \leqslant CK^C$, or $|A| > |\mathbf{G}(q)|/CK^C$, where $C = C(r) > 0$ is a constant depending on the rank r of $\mathbf{G}$ only.

To translate between the two formulations: take $K = |A|^\varepsilon$ and apply Tao's lemma relating small doubling and approximate groups.

Some consequences: diameter bounds

Babai's conjecture for bounded rank finite simple groups:
$\exists C = C(r) > 0$ s.t.

$$diameter(\mathbf{G}(q)) \leqslant (\log |\mathbf{G}(q)|)^C$$

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Indeed: if S is a generating set, applying the product theorem to $A := S^{3^n}$ repeatedly yields $|S^{3^n}| > |S|^{(1+\varepsilon)^n}$ unless $S^{3^n} = \mathbf{G}(q)$.

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Note however that for bounded rank much more is expected:

Conjecture (Diameter bound for bounded rank groups)

$\exists C = C(r) > 0$ such that if $\mathbf{G}(q)$ is a finite simple group with rank at most r,

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$\rightarrow$ known to hold (Breuillard-Gamburd) only for $\mathbf{G}(q) = PSL_2(p)$ and only for a set of primes of full density (among all primes).

more consequences: infinite groups

Recall the *strong-approximation theorem* of Matthews-Vaserstein-Weisfeiler: if $\Gamma \leqslant \mathbf{G}(\mathbb{Q})$ is a finitely generated Zariski-dense subgroup and $\mathbf{G}$ a simply connected semisimple algebraic group, then for almost all primes p the reduction mod p map $\mathbf{G}(\mathbb{Z}) \to \mathbf{G}(\mathbb{Z}/p\mathbb{Z})$ is surjective in restriction to Γ.

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Corollary

Let $\mathbf{G}$ *be a semisimple algebraic group. Then* $\exists \varepsilon = \varepsilon(dim(\mathbf{G})) > 0$ *s.t. if* A *is any finite subset of* $\mathbf{G}(\mathbb{C})$ *generating a Zariski dense subgroup*

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Proof sketch: $\Gamma := \langle A \rangle$ maps onto $\mathbf{G}(\mathbb{F}_p)$ for infinitely many primes p. Apply the product theorem to the image of A.

Proof of the product theorem: 1) the Larsen-Pink non-concentration estimate

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This fails of course for subgroups of $\mathrm{GL}_n(\mathbb{F}_q)$. Brauer and Feit, then Weisfeiler gave characteristic p versions of Jordan's theorem culminating, by means of CFSG, in the full elucidation of the subgroup structure of $\mathrm{GL}_n(\mathbb{F}_q)$.

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The Larsen-Pink result essentially says that every subgroup Γ of $\mathbf{G}(\mathbb{F}_q)$ ($\mathbf{G}$=simple algebraic group over $\mathbb{F}_q$) is (a conjugate of) $\mathbf{G}(\mathbb{F}'_q)$ for some smaller field $\mathbb{F}'_q \leqslant \mathbb{F}_q$, unless it is contained in a proper *algebraic* subgroup of $\mathbf{G}$.

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A key step in their proof consists in showing the following *non-concentration estimate*:

Theorem (Larsen-Pink non-concentration estimate)

Suppose Γ is a subgroup of $\mathbf{G}(\mathbb{F}_q)$ which is "sufficiently Zariski-dense" in $\mathbf{G}$, then for every algebraic subvariety $\mathcal{V} \leqslant \mathbf{G}$ we have:

$$|\Gamma \cap \mathcal{V}| < C_{\mathcal{V}} |\Gamma|^{\frac{\dim \mathcal{V}}{\dim \mathbf{G}}}$$

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In the BGT proof of the product theorem, the first step consists in adapting the above to approximate subgroups:

Theorem (Larsen-Pink for approximate subgroups)

Suppose A *is a* K*-approximate subgroup of* $\mathbf{G}(\mathbb{F}_q)$ *which is "sufficiently Zariski-dense" in* $\mathbf{G}$*, then for every algebraic subvariety* $\mathcal{V} \leqslant \mathbf{G}$ *we have:*

$$|A \cap \mathcal{V}| < K^{C_{\mathcal{V}}} |A|^{\frac{\dim \mathcal{V}}{\dim \mathbf{G}}}$$

Proof of the product theorem: 2) counting tori

The argument of the BGT proof then runs roughly as follows:

- take $\mathcal{V} :=$ non regular semisimple elements and apply Larsen-Pink $\rightarrow$ get that most elements of A are regular semisimple.
- if $a \in A^2$ is regular semisimple with centralizer $C(a)$, then applying Larsen-Pink to both $\mathcal{V} := C(a)$ and $\mathcal{V} := \{gag^{-1}, g \in \mathbf{G}(\mathbb{F}_q)\}$ and using the (approximate) orbit-stabilizer formula, we get (T denotes a maximal torus).

$$|A^2 \cap C(a)| \simeq |A|^{\frac{\dim T}{\dim \mathbf{G}}}$$

.

- This means that whenever A^2 intersects a maximal torus T non trivially (i.e. contains a regular element), the intersection must be large i.e. $\geqslant |A|^{\frac{\dim T}{\dim \mathbf{G}}}$.
- This implies that the set of tori intersecting A^2 non trivially is stable under conjugation by A, hence by $\langle A \rangle = \mathbf{G}(\mathbb{F}_q)$, hence contains all tori, and it follows that A is almost all of $\mathbf{G}(q)$.

towards a complete classification of approximate subgroups of linear groups

The product theorem was stated for generating subsets of simple groups. It does not holds without modification for generating subsets of other groups (even semisimple). One conjectures the following:

Conjecture (arbitrary linear approximate subgroups)

$\exists C = C(d) > 0$ *s.t. if* $A \subset \mathrm{GL}_d(\mathbb{F}_q)$ *is a* K*-approximate subgroup, then there are subgroups* $N \lhd H$ *normalised by* A *with* $N \subset A^C$ *and* H/N *nilpotent such that* A *is contained in* $\leqslant K^C$ *cosets of* H.

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other open pb: get good explicit estimates on ε in the product theorem!