Approximate groups and their applications: part 4

E. Breuillard

Université Paris-Sud, Orsay

St. Andrews, August 3-10, 2013

In the classical setting, one studies the *growth of a f.g. group* by fixing a finite generating set S, and by asking for the asymptotics of

$$|S^n|$$

as n tends to $+\infty$.

In the classical setting, one studies the *growth of a f.g. group* by fixing a finite generating set S, and by asking for the asymptotics of

$$|S^n|$$

as n tends to $+\infty$.
In the theory of *approximate groups*, one does not fix S but considers instead a large finite set A and asks how the size of

$$|A^n|$$

compares to that of A for small values of n, typically $n = 2, 3$.

Here is nice result, which combines the two aspects:

Theorem (Razborov, Safin)

Let A be a finite subset of a free group not contained in a cyclic subgroup. Then for every $n \geqslant 3$,

$$|A^n| \geqslant \frac{|A|^{\frac{n+1}{2}}}{100^n}$$

In particular $|AAA| \geqslant c|A|^2$.

Approximate subgroups in infinite groups

Here is nice result, which combines the two aspects:

Theorem (Razborov, Safin)

Let A be a finite subset of a free group not contained in a cyclic subgroup. Then for every $n \geqslant 3$,

$$|A^n| \geqslant \frac{|A|^{\frac{n+1}{2}}}{100^n}$$

In particular $|AAA| \geqslant c|A|^2$.

Remark: for word hyperbolic groups, one can prove (BGT) that K-approximate groups are $O_K(1)$ covered by a cyclic subgroup, but the bounds on $O_K(1)$ are far from optimal.

As it turns out, approximate groups were already hidden in Gromov's original paper on polynomial growth. Recall his celebrated theorem:

Theorem (Gromov 1982)

Every finitely generated group with polynomial growth is virtually nilpotent.

virtually nilpotent = there is a finite index subgroup which is nilpotent.

polynomial growth = $|S^n| = O(n^C)$ for some $C > 0$.

How is Gromov's theorem related to approximate groups ?

How is Gromov's theorem related to approximate groups ?
Let G be a group with polynomial growth:

$$|S^n| \leqslant n^C \text{ for all large } n.$$

At the start of the proof, Gromov argues that there are infinitely many integers n_k's for which:

$$|S^2_{n_k}| \leqslant K|S_{n_k}|$$

for some $K > 0$ ($K = 3^C$ works).

How is Gromov's theorem related to approximate groups ?

Let G be a group with polynomial growth:

$$|S^n| \leqslant n^C \text{ for all large } n.$$

At the start of the proof, Gromov argues that there are infinitely many integers n_k's for which:

$$|S_{n_k}^2| \leqslant K|S_{n_k}|$$

for some $K > 0$ ($K = 3^C$ works).

Indeed, otherwise $|S^{2n}| > 3^C|S^n|$ for all large n. Hence $|S^{2^n}| > 3^{Cn} > (2^n)^C$ contradicting the assumption.

Main theorem, weak form

G = a group.
$A \subset G$ a finite subset.

Theorem (BGT 2012 weak form)

Assume $|AA| \leqslant K|A|$. Then there is a virtually nilpotent subgroup $\Gamma \leqslant G$ and $g \in G$ such that

$$|A \cap g\Gamma| \geqslant |A|/O_K(1).$$

Gromov's theorem on groups of polynomial growth.

From our theorem, we recover Gromov's theorem!

Theorem (Gromov 1982)

Every group with polynomial growth is virtually nilpotent.

From our theorem, we recover Gromov's theorem!

Theorem (Gromov 1982)

Every group with polynomial growth is virtually nilpotent.

Suppose $|S^r| \leqslant r^C$ for all large integers r.

- There are arbitrarily large scales r such that
$$|S^{2r}| \leqslant 3^C |S^r|.$$
- By the theorem applied to $A := S^r$ we get that S^{2r} intersects a virtually nilpotent group Γ in a set of size $\geqslant |S^r|/O_C(1)$
- apply Ruzsa's covering lemma to get that $[G : \Gamma]$ must be finite.

Gromov's theorem on groups of polynomial growth.

From our theorem, we recover Gromov's theorem!

Theorem (Gromov 1982)

Every group with polynomial growth is virtually nilpotent.

Suppose $|S^r| \leqslant r^C$ for all large integers r.

- There are arbitrarily large scales r such that

$$|S^{2r}| \leqslant 3^C |S^r|.$$

- By the theorem applied to $A := S^r$ we get that S^{2r} intersects a virtually nilpotent group Γ in a set of size $\geqslant |S^r|/O_C(1)$
- apply Ruzsa's covering lemma to get that $[G : \Gamma]$ must be finite.

Remark: the argument works assuming only that $|B(r)| \leqslant r^C$ for one large r.

Approximate groups, definition

Recall our definition. Let $K \geqslant 1$ be a parameter.

Definition (Approximate subgroup)

A (finite) subset A in an ambient group G, is called a K-approximate subgroup if:

- $A = A^{-1}$ and $1 \in A$,
- $AA \subset XA$ for some subset $X \subset G$ with $|X| \leqslant K$.

Approximate groups, definition

Recall our definition. Let $K \geqslant 1$ be a parameter.

Definition (Approximate subgroup)

A (finite) subset A in an ambient group G, is called a K-approximate subgroup if:

- $A = A^{-1}$ and $1 \in A$,
- $AA \subset XA$ for some subset $X \subset G$ with $|X| \leqslant K$.

Recall also the following important fact: if A is any set with $|AAA| \leqslant K|A|$, then $B := (A \cup A^{-1} \cup \{1\})^3$ is a K^C-approximate subgroup with $|B| \leqslant K^C|A|$, and C is an absolute constant (indept. of K).

Examples of approximate groups

- a finite group is a 1-approximate group.
- a d-dimensional progression is a 2^d-approximate group.
- a small ball around the identity in a Lie group (not a finite approximate group though!).
- a nilprogression of rank r and step s is a $O_{r,s}$-approximate group.
- "extensions" of such (the so-called *coset nilprogressions*).

What is a nilprogression ?

What is a nilprogression ?

"Nilprogression" $=$ a homomorphic image $P = \pi(B)$ of a "box" B in the free nilpotent group $N_{r,s}$ of rank r and step s under a group homomorphism $\pi : N_{r,s} \to G$.

Examples of approximate groups

What is a nilprogression ?

"Nilprogression" $=$ a homomorphic image $P = \pi(B)$ of a "box" B in the free nilpotent group $N_{r,s}$ of rank r and step s under a group homomorphism $\pi : N_{r,s} \to G$.

"Box" means: a set of the form

$$\{e_1^{x_1} \cdot \ldots \cdot e_r^{x_r} \cdot \prod_c c^{y_c}\}$$

where the e_i's are generators of the free nilpotent group, the c's form a (Hall) basis of basic commutators, and

$$|x_i| \leqslant N_i, |y_c| \leqslant N(c) := \prod_{e_i \in c} N_i.$$

Examples of approximate groups

What is a nilprogression ?

"Nilprogression" $=$ a homomorphic image $P = \pi(B)$ of a "box" B in the free nilpotent group $N_{r,s}$ of rank r and step s under a group homomorphism $\pi : N_{r,s} \to G$.

"Box" means: a set of the form

$$\{e_1^{x_1} \cdot \ldots \cdot e_r^{x_r} \cdot \prod_c c^{y_c}\}$$

where the e_i's are generators of the free nilpotent group, the c's form a (Hall) basis of basic commutators, and

$$|x_i| \leqslant N_i, |y_c| \leqslant N(c) := \prod_{e_i \in c} N_i.$$

The N_i's are called the "side-lengths" of the nilprogression.

What is a nilprogression ?

"Nilprogression" $=$ a homomorphic image $P = \pi(B)$ of a "box" B in the free nilpotent group $N_{r,s}$ of rank r and step s under a group homomorphism $\pi : N_{r,s} \to G$.

Example: If $N_1, N_2 \in \mathbb{N}$, set

$$A := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} ; |x| \leqslant N_1; |y| \leqslant N_2; |z| \leqslant N_1 N_2 \right\}$$

It is a nilprogression of step 2 and rank 2.

Freiman's theorem and its generalizations

Recall the statement of Freiman's theorem, as generalized by Green and Ruzsa to an arbitrary *abelian* group:

Theorem (Green-Ruzsa 2006)

Suppose G is an abelian group and $A \subset G$ a K-approximate subgroup. Then

$$A \subset P \subset X + A,$$

where

- $|X| = O_K(1)$,
- *P is coset multi-dimensional progression P of dimension $d = O_K(1)$.*
- $|P| \leqslant O_K(1)|A|$.

Freiman's theorem and its generalizations

Recall the statement of Freiman's theorem, as generalized by Green and Ruzsa to an arbitrary *abelian* group:

Theorem (Green-Ruzsa 2006)

Suppose G is an abelian group and $A \subset G$ a K-approximate subgroup. Then

$$A \subset P \subset X + A,$$

where

- $|X| = O_K(1)$,
- *P is coset multi-dimensional progression P of dimension $d = O_K(1)$.*
- $|P| \leqslant O_K(1)|A|$.

Remark: A coset multi-dimensional progression is a set of the form $P = H + L$, where $H \leqslant G$ is a finite subgroup and L is a multi-dimensional progression (i.e. $L = \pi(\prod_1^d [-N_i, N_i])$).

Approximate subgroups of nilpotent groups

Breuillard-Green (2009) extended Freiman's theorem to torsion-free nilpotent groups. Recently Matt Tointon treated the general nilpotent case and obtained the following extension of Freiman's theorem to an arbitrary nilpotent group:

Theorem (M. Tointon 2013)

There is $C = C_s > 0$ s.t. if A is a K-approximate subgroup of a nilpotent group of nilpotency class $\leqslant s$, then there is a coset nilprogression P of rank $\leqslant K^{C_s}$ such that

$$A \subset P \subset A^{K^{C_s}} \subset YA$$

for some Y of size $\leqslant \exp(K^{C_s+1})$.

Remark: A coset nilprogression is a set of the form $P = HL$, where $H \leqslant G$ is a finite subgroup normalized by the set L, such that $H \backslash HL$ is a nilprogression (in other words: it is the inverse image of a nilprogression by a homomorphism with finite kernel).

Main theorem, strong form

Let G be a group and $K \geqslant 1$ a parameter.

Theorem (BGT strong form: structure of approximate groups)

Let $A \subset G$ be a finite K-approximate subgroup. Then

$$A \subset XP,$$

where

- $|X| \leqslant O_K(1)$,
- *P is a coset nilprogression of rank and step $O_K(1)$,*
- $P \subset A^4$.

coset-nilprogression = inverse image of a nilprogression by a homomorphism with finite kernel.

Proof of the structure theorem

The proof is contained in the paper:

E. Breuillard, B. Green and T. Tao, *The structure of approximate groups*, Publ. Inst. Hautes Études Sci., **116**, Issue 1, 115–221, (2012)

and can be seen as a continuation of the ideas developed by Hrushovski in

E. Hrushovski, *Stable group theory and approximate subgroups*, J. Amer. Math. Soc. 25 (2012), no. 1, 189–243.

Proof of the structure theorem

The proof is contained in the paper:

- E. Breuillard, B. Green and T. Tao, *The structure of approximate groups*, Publ. Inst. Hautes Études Sci., **116**, Issue 1, 115–221, (2012)

and can be seen as a continuation of the ideas developed by Hrushovski in

- E. Hrushovski, *Stable group theory and approximate subgroups*, J. Amer. Math. Soc. 25 (2012), no. 1, 189–243.

The proof is close in spirit to Gromov's original proof. It makes use of ultrafilters to build so-called *K-ultra-approximate groups*, i.e. ultraproducts of K-approximate groups. The heart of the proof lies in an transposition to the approximate group setting of the group theoretical arguments used in the Gleason-Yamabe and Montgomery-Zippin proof of the structure theorem for locally compact groups in relation to Hilbert 5-th problem.

An application to geometry: almost non-negatively curved manifolds

In 1978 Gromov proved that almost flat manifolds are finitely covered by nilmanifolds, in particular they have virtually nilpotent fundamental group. This last fact was later generalized by Fukaya-Yamaguchi, then Cheeger-Colding, Kapovitch-Wilking, to almost non-negatively curved (sectional or Ricci) manifolds. We recover this:

Corollary

There is $\varepsilon = \varepsilon(n) > 0$ such that every closed n-manifold with diameter 1 and Ricci curvature $\geqslant -\varepsilon$ has virtually nilpotent π_1.

Applications: diameter of finite groups

Here is another application of the main theorem, this time to finite groups!

$$G = \text{ a finite group,}$$
$$\mathcal{G} = \text{ its Cayley graph w.r.t some generating set.}$$
$$D = diam(\mathcal{G}) \text{ the diameter of } \mathcal{G}.$$

Applications: diameter of finite groups

Here is another application of the main theorem, this time to finite groups!

$$G = \text{ a finite group},$$
$$\mathcal{G} = \text{ its Cayley graph w.r.t some generating set.}$$
$$D = diam(\mathcal{G}) \text{ the diameter of } \mathcal{G}.$$

Theorem (Structure of large diameter groups)

$\forall \varepsilon > 0$, $\exists C = C(\varepsilon) > 0$ *s.t. if* $D \geqslant |G|^{\varepsilon}$, *there is*

- *A subgroup* $G_0 \leqslant G$ *of index* $\leqslant C$,
- *A normal subgroup* H *of* G_0 *s.t.* G_0/H *is nilpotent with of nilpotency class* $\leqslant C$ *and number of generators* $\leqslant C$ *and*

$$|G_0/H| \geqslant |G|^{\varepsilon/2}$$

More applications: diameter of finite groups

Corollary (Diameter of finite simple groups)

$\forall \varepsilon > 0 \; \exists C_\varepsilon > 0$ *s.t.*

$$diam(\mathcal{G}) \leqslant C_\varepsilon |G|^\varepsilon,$$

for every Cayley graph $\mathcal{G}$ of every (non-abelian) finite simple group G.

More applications: diameter of finite groups

Corollary (Diameter of finite simple groups)

$\forall \varepsilon > 0 \ \exists C_\varepsilon > 0$ *s.t.*

$$diam(\mathcal{G}) \leqslant C_\varepsilon |G|^\varepsilon,$$

for every Cayley graph $\mathcal{G}$ of every (non-abelian) finite simple group G.

Babai's conjecture $= \exists C > 0$ s.t.

$$diam(\mathcal{G}) \leqslant C(\log |G|)^C,$$

for some absolute constant $C > 0$.

Corollary (Diameter of finite simple groups)

$\forall \varepsilon > 0 \ \exists C_\varepsilon > 0$ *s.t.*

$$diam(\mathcal{G}) \leqslant C_\varepsilon |G|^\varepsilon,$$

for every Cayley graph $\mathcal{G}$ of every (non-abelian) finite simple group G.

Babai's conjecture $= \exists C > 0$ s.t.

$$diam(\mathcal{G}) \leqslant C(\log|G|)^C,$$

for some absolute constant $C > 0$.

Our proof does not use CFSG !

For Cayley graphs, an lower bound on the diameter yields a lower bound on the λ_1. Indeed:

More applications: lower bounds on λ_1

For Cayley graphs, an lower bound on the diameter yields a lower bound on the λ_1. Indeed:

Lemma (Diaconis-Saloff-Coste)

If $\mathcal{G}$ is a k-regular Cayley graph of a finite group G and $D(\mathcal{G})$ denotes its diameter, then:

$$\lambda_1(\mathcal{G}) \geqslant \frac{2}{kD(\mathcal{G})^2}$$

For Cayley graphs, an lower bound on the diameter yields a lower bound on the λ_1. Indeed:

Lemma (Diaconis-Saloff-Coste)

If $\mathcal{G}$ is a k-regular Cayley graph of a finite group G and $D(\mathcal{G})$ denotes its diameter, then:

$$\lambda_1(\mathcal{G}) \geqslant \frac{2}{kD(\mathcal{G})^2}$$

More applications: lower bounds on λ_1

We thus get:

Corollary (Uniform lower bound on λ_1 for FSG)

Given any $\varepsilon > 0$ and $k > 0$, every k-regular Cayley graph of every non-abelian finite simple group G satisfies:

$$\lambda_1(\mathcal{G}) \geqslant \frac{1}{|G|^{\varepsilon}},$$

provided $|G| > f(\varepsilon, k)$.

Remarks:

- For bounded rank FSG, the product theorem yields the better bound: $\lambda_1(\mathcal{G}) \geqslant 1/k(\log|G|)^{C(r)}$.
- Our bound applies with no restriction and the rank.
- It is independent of CFSG.