Orbit coherence in permutation groups

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Orbit partitions

Let G be a group of permutations of a set Ω.

Definitions

- For $g \in G$, write $\pi(g)$ for the partition of Ω given by the orbits of g.
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Example

$$\pi(S_3) = \Big\{ \{\{1\},\{2\},\{3\}\},\ \{\{1,2\},\{3\}\},\ \{\{1,3\},\{2\}\},\ \{\{1\},\{2,3\}\},\ \{\{1,2,3\}\} \Big\}.$$

The partition lattice

Let ρ and σ be partitions of Ω. Say ρ is a *refinement* of σ if every part of σ is a union of parts of ρ.

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The set $P(\Omega)$ is a lattice under the refinement order. Any two partitions ρ and σ have

- a greatest common refinement $\rho \wedge \sigma$ (their *meet*).
- a least common coarsening $\rho \vee \sigma$ (their *join*).

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- a chain;
- a sublattice of $P(\Omega)$;
- a lower subsemilattice (*meet-coherence*);
- an upper subsemilattice (*join-coherence*).

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An ingredient in the proof of the last part is that a group acting regularly is join-coherent if and only if it is *locally cyclic*.

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In particular, centralizers in S_n always give sublattices.

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Corollary

For $i \in \mathbb{N}$ let G_i be a join-coherent permutation group on the finite set Ω_i. Then the profinite wreath product $\cdots \wr G_2 \wr G_1$ is join-coherent on $\prod_{i \in \mathbb{N}} \Omega_i$.

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The finite primitive join-coherent groups are

- S_n *in its natural action;*
- *transitive subgroups of* $\mathrm{AGL}_1(p)$.

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Let G be a permutation group on n points which normalizes an n-cycle. Let n have prime factorization $\prod_i p_i^{a_i}$. Then G is join-coherent if and only if it is isomorphic to $\prod_i G_i$, where G_i is a transitive permutation group on $p_i^{a_i}$ points, the orders of the groups G_i are mutually coprime, and one of the following holds for each i:

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- *G_i is cyclic of order $p_i^{a_i}$,*
- *$a_i = 1$ and G_i is a transitive subgroup of $\mathrm{AGL}_1(p_i)$,*
- *$a_i > 1$ and G_i is the extension of a cyclic group of order $p_i^{a_i}$ by the automorphism $x \mapsto x^r$, where $r = p_i^{a_i-1} + 1$.*