The 6-transposition quotients of the Coxeter groups $G^{(m,n,p)}$

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Outline

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Which groups have property (σ)?

Property (σ)

A group G satisfies (σ) if it satisfies two conditions:

(i) *G is generated by three involutions $a,\ b,\ c$ two of which commute, say $ab = ba$;*

(ii) *for all $t,\ s \in T := a^G \cup b^G \cup (ab)^G \cup c^G$ the product ts has order at most 6.*

G has $(\sigma) \Rightarrow G$ is a quotient of a Coxeter group $G^{(m,n,p)}$ for $m,\ n,\ p \in [1,6]$:

$$G^{(m,n,p)} := \langle a, b, c \mid a^2,\ b^2,\ c^2,\ (ab)^2,\ (ac)^m,\ (bc)^n,\ (abc)^p \rangle.$$

Moreover which of the groups having (σ) embed in $\mathbb{M}$, the Monster simple group, such that a, b, ab and c are mapped to the conjugacy class $2A$ of $\mathbb{M}$?

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(ii) *for all t, $s \in T := a^G \cup b^G \cup (ab)^G \cup c^G$ the product ts has order at most 6.*

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Motivation

Original goal:

- Classify all Majorana algebras generated by three axes $\langle\langle a_a, a_b, a_c \rangle\rangle$ such that the subalgebra $\langle\langle a_a, a_b \rangle\rangle$ is of *type 2A*,
- which of these are subalgebras of $V_{\mathbb{M}}$, the Monster algebra?

Majorana theory: axiomatisation by A. A. Ivanov of 7 of the properties of $V_{\mathbb{M}}$ and of some of its idempotents called $2A$-axes.
Two distinct objectives:

- describe a class of algebras independently of $\mathbb{M}$,
- describe subalgebras of $V_{\mathbb{M}}$ using the subgroup structure of $\mathbb{M}$.

Proposition (Conway, 1984)

There is a bijection ψ between the $2A$-involutions of $\mathbb{M}$ and the $2A$-axes of $V_{\mathbb{M}}$.

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For a finite group G a Majorana representation is a tuple:

$$\mathcal{R} = (G,\ T,\ X,\ (\ ,\ ),\ \cdot,\ \phi,\ \psi)$$

- T is a G-invariant set of involutions generating G,
- $(X, (\ ,\ ), \cdot)$ is an algebra satisfying **($M1$)**and **($M2$)**,
- $\phi : G \to Aut(X)$ is a representation of G with kernel $Z(G)$,
- $\psi : T \hookrightarrow A_T$ sends each $t \in T$ to a **Majorana axis $a_t := \psi(t)$** of X , such that $\phi(t)$ acts on X as the **Majorana involution** $\tau(\psi(t))$; $\forall g \in G\ a_{t^g} = a_t^{\phi(g)}$),
- and lastly we require that $\psi(T)$ generates X.

We call X a *Majorana algebra* for G and write $X = \langle\langle A \rangle\rangle$ for $A := \{a_t\}_{t \in T}$.

Example

$\mathcal{R} = (\mathbb{M}, 2A, V_{\mathbb{M}}, (\ ,\ ), \cdot, \psi)$ is a Majorana representation of $\mathbb{M}$ with Majorana algebra $V_{\mathbb{M}}$.

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Example

For any subgroup H of $\mathbb{M}$ generated by a H-invariant set of $2A$-involutions, one can define a Majorana algebra for H which is a subalgebra $V_{\mathbb{M}}$.

First look at *dihedral subalgebras* of $V_{\mathbb{M}}$.

Lemma (The 6-transposition property)

For $t,\ s \in 2A$ the product ts belongs to either of the $\mathbb{M}$ conjugacy classes: $1A,\ 2A,\ 2B,\ 3A,\ 3C,\ 4A,\ 4B,\ 5A,\ 6A$.

Theorem (Conway, Norton, 1985)

For any $t, s \in 2A$ there are 9 isomorphism types of dihedral subalgebras $\langle\langle\psi(t), \psi(s)\rangle\rangle$ in $V_{\mathbb{M}}$.

Theorem (A. A. Ivanov et al, 2009)

There are exactly 9 dihedral Majorana algebras obtained from the dihedral groups and they are equal to the dihedral subalgebras of $V_{\mathbb{M}}$.

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Corollary

For Let G be a finite group. If G has a Majorana representation $\mathcal{R} = (G,\ T,\ X,\ (\ ,\ ),\ \cdot,\ \phi,\ \psi)$ then the involutions in T are 6-transpositions.

Let G be a group generated by 3 involutions a, b, c with $ab = ba$, and define a Majorana representation of G with:

- $T = a^G \cup b^G \cup (ab)^G \cup c^G$;
- $X = \langle\langle \psi(a), \psi(b), \psi(c) \rangle\rangle$;
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What are the possible groups G? They must satisfy (σ).
For which such groups G is X a subalgebras of $V_{\mathbb{M}}$? The groups G must embed in $\mathbb{M}$ such that a, b, c, ab are mapped to $2A$-involutions.

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- the subalgebra $\langle\langle \psi(a), \psi(b) \rangle\rangle$ has type $2A$.

What are the possible groups G? They must satisfy (σ).
For which such groups G is X a subalgebras of $V_{\mathbb{M}}$? The groups G must embed in $\mathbb{M}$ such that a, b, c, ab are mapped to $2A$-involutions.

Property (σ)

A group G satisfies (σ) if it satisfies two conditions:

(i) *G is generated by $a,\ b,\ c$, or order dividing 2, two of which commute, say $ab = ba$;*

(ii) *for all $t,\ s \in T := a^G \cup b^G \cup (ab)^G \cup c^G$ the product ts has order at most 6.*

Theorem (D. 2013)

A group has property (σ) if and only if it is a quotient of at least one of the following 11 finite groups:

1)	$2 \text{ wr } 2^2$	7)	$2^4 :_{\lambda_2} A_5$
2)	$(S_3 \times S_3) : 2^2$	8)	$2 \times S_6$
3)	$2^4 : D_{10}$	9)	$(2^4 : (S_3 \times S_3)) \times 2$
4)	$2 \times S_5$	10)	$2^5 :_{\phi} S_5$
5)	$L_2(11)$	11)	$(3^4 : 2) : (3_+^{1+2} : 2^2)$
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Group	Isomorphism Type	Quotient of $G^{(m,n,p)}$ for $(m,n,p) =$	Centre Order	Subgroups
G_1	$2 \; wr \; 2^2$	$(4,4,4)$	2	
G_2	$(S_3 \times S_3) : 2^2$	$(4,4,6)$	2	
G_3	$2^4 : D_{10}$	$(4,5,5)$	1	
G_4	$2 \times S_5$	$(4,5,6)$	2	
G_5	$L_2(11)$	$(5,5,5)$	1	
G_6	$(2^4 :_{\phi_1} D_{12}) \times 2$	$(4,6,6)$	2	
G_7	$2^4 :_{\lambda_2} A_5$	$(6,5,5)$	1	G_3
G_8	$2 \times S_6$	$(6,6,5)$	2	$G_2, \; G_4$
G_9	$\left(2^4 : (S_3 \times S_3)\right) \times 2$	$(6,6,6)$	2	
G_{10}	$2^5 :_{\phi} S_5$	$(6,6,6)$	2	$G_3, \; G_4, \; G_6$
G_{11}	$(3^4 : 2) : (3_+^{1+2} : 2^2)$	$(6,6,6)$	1	

Outline

Which of the G_i's embed into $\mathbb{M}$ such that $a,\ b,\ ab,\ c$ are mapped to class $2A$ of $\mathbb{M}$?

From a result of S. Norton we obtain:

Proposition (Norton, 1985)

Except for G_9 and G_{11} all the groups G_i $2A$-embed into $\mathbb{M}$. Moreover the largest quotients of G_9 and G_{11} which $2A$-embed into $\mathbb{M}$ are:

- $G_9/Z(G_9) \cong 2^4 : (S_3 \times S_3)$;
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Outline

1. Introduction
   - Property (σ)
2. Motivation
   - Majorana representation
   - Dihedral subalgebras
3. Main theorem
   - Norton's embeddings
4. Proof
   - The finite cases
   - The infinite cases $(m, n, p) \neq (6, 6, 6)$
   - The infinite case $(m, n, p) = (6, 6, 6)$

The finite $G^{(m,n,p)}$ groups

We assume $2 \le m \le n \le p \le 6$ wlog.

Theorem (Coxeter 1939, Edjvet 1994)

$2 \le m \le n \le p \le 6$ then the group $G^{(m,n,p)}$ is finite if and only if

$$(m, n, p) \notin \{(4, 6, 6),\ (5, 5, 6),\ (5, 6, 6),\ (6, 6, 6)\}.$$

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$G^{(3,5,5)} \cong A_5, \quad G^{(3,6,6)} \cong 3_+^{1+2} : 2^2, \quad G^{(5,5,5)} = L_2(11).$

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Outline

Introduce the relation $R_1^{r_1} = 1$ for $R_1 = a \cdot b^c = acbc$, where $r_1 \in [1, 6]$. Denote $G^{(m,n,p:r_1)}$ the quotient of $G^{(m,n,p)}$ by the normal closure of $R_1^{r_1}$;

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Let us describe the isomorphism types of the groups $G^{(m,n,p;\ r_1)}$.

Definition

We say that $G^{(m,n,p;\ r_1)}$ does not *shrink* if the orders of ac, bc and abc are not smaller than m, n and p respectively.

Example

Let $G := G^{(4,6,6;4)}$. Magma gives $|G| = 192$. Let N be the normal closure of $\langle a \rangle$. Now $G/N = \langle b, c \rangle \cong D_{12}$ so that $|N| = 32$. We can check that $N = \langle a, a^c, a^{cb}, a^{cbc}, a^{cbcb} \rangle \cong 2^5$, so that $G = N : H$, where action of H on N gives $Z(G) = \langle aa^{cbc}a^{cbcb} \rangle \cong 2$.

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The groups $G^{(m,n,p:\ r_1)}$ which do not shrink are as follows:

Assume $(m, n, p) \in \mathcal{S}$ and $r_1 \in [1, 6]$.
Let us describe the isomorphism types of the groups $G^{(m,n,p:\ r_1)}$.

Definition

We say that $G^{(m,n,p:\ r_1)}$ does not *shrink* if the orders of ac, bc and abc are not smaller than m, n and p respectively.

Example

Let $G := G^{(4,6,6;4)}$. Magma gives $|G| = 192$. Let N be the normal closure of $\langle a \rangle$. Now $G/N = \langle b, c \rangle \cong D_{12}$ so that $|N| = 32$. We can check that $N = \langle a, a^c, a^{cb}, a^{cbc}, a^{cbcb} \rangle \cong 2^5$, so that $G = N : H$, where action of H on N gives $Z(G) = \langle aa^{cbc}a^{cbcb} \rangle \cong 2$.

Proposition

The groups $G^{(m,n,p:\ r_1)}$ which do not shrink are as follows:

$(m, n, p;\ r_1)$	Iso. Type	$(\boldsymbol{\sigma})$	Element contradicting $(\boldsymbol{\sigma})$
$(4, 6, 6;\ 4)$	$2 \times (2^4 :_{\phi_1} D_{12})$	**Y**	–
$(6, 6, 4;\ 3)$	$2^4 :_{\phi_1} D_{12}$	**Y**	–
$(6, 6, 4;\ 6)$	$2^2.2 \times (2^4 :_{\phi_1} D_{12})$	N	$ab \cdot a^c$ has order 8
$(6, 5, 5;\ 5)$	$2^4 :_{\lambda_2} A_5$	**Y**	–
$(5, 5, 6;\ 3)$ $\cong (3, 5, 10)$	$2 \times A_5$	N	$ab \cdot a^c$ has order 10
$(5, 5, 6;\ 6)$	$2.(2^4 :_{\lambda_2} A_5)$	N	$ab \cdot a^c$ has order 10

$(m, n, p;\ r_1)$	Iso. Type	$(\boldsymbol{\sigma})$	Element contradicting $(\boldsymbol{\sigma})$
$(6, 6, 5;\ 4)$	$2 \times S_6$	**Y**	–
$(6, 6, 5;\ 5)$	$2 \times L_2(11)$	N	$ab \cdot a^c$ has order 10
$(5, 6, 6;\ 5)$	$2 \times L_2(11)$	N	$ab \cdot a^c$ has order 10
$(5, 6, 6;\ 6)$	$(2^3 : 3) : (2 \times S_6)$	N	$ab \cdot a^c$ has order 12

Outline

For $(m,n,p) = (6,6,6)$ we introduce four relations $R_i^{r_i} = 1$:

$$R_1^{r_1} = (a \cdot b^c)^{r_1},\ R_2^{r_2} = (ab \cdot a^c)^{r_2},\ R_3^{r_3} = (ab \cdot b^c)^{r_3},\ R_4^{r_4} = (c \cdot b^{ca})^{r_4},$$

where $r_i \in [1,6]$ for all i.

$$G^{(m,n,p:\ r_1,\ r_2,r_3,r_4)} :=$$
$$\langle a,b,c \mid a^2,\ b^2,\ c^2,\ (ab)^2,\ (ac)^m,\ (bc)^n,\ (abc)^p,\ R_1^{r_1},\ R_2^{r_2},\ R_3^{r_3},\ R_4^{r_4} \rangle.$$

Proposition (Magma)

The groups $G^{(m,n,p:\ r_1,\ r_2,r_3,r_4)}$ are finite for $r_1, r_2, r_3, r_4 \in [1,6]$.

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Proposition

Let $G := G^{(6,6,6;\ r_1,\ r_2,\ r_3)}$.

(i) *If* $1 \in \{r_1,\ r_2,\ r_3\}$ *then* G *is a quotient of the group* 2^2;

(ii) *If* $2 \in \{r_1,\ r_2,\ r_3\}$ *then* G *is a quotient of the group* $S_3 \times S_3 \times 2$;

(iii) *If* $3 \in \{r_1,\ r_2,\ r_3\}$ *then* G *is a quotient of the group* D_{12};

(iv) *If* $4 \in \{r_1,\ r_2,\ r_3\}$ *then* G *is a quotient of the group* $\left(2^4 : (S_3 \times S_3)\right) \times 2$;

(v) *If* $\{5,\ 6\} \subseteq \{r_1,\ r_2,\ r_3\}$ *then* G *is a quotient of the group* 2^2.

Lemma

All the groups above satisfy (σ) .

Remark

Only cases left: (r_1, r_2, r_3) *equal to* $(5, 5, 5)$ *or* $(6, 6, 6)$.

Proposition

Let $(m, n, p) = (6, 6, 6)$.

- *for* $(r_1,\ r_2,\ r_3) = (5, 5, 5)$ *the largest quotient of* $G^{(6,6,6;\ 5,5,5)}$ *satisfying* (σ) *is* $2^5 : S_5$;
- *for* $(r_1,\ r_2,\ r_3) = (6, 6, 6)$ *the largest quotient of* $G^{(6,6,6;\ 6,6,6)}$ *satisfying* (σ) *is* $(3^4 : 2) : (3_+^{1+2} : 2^2)$.

What next?

Classify all the Majorana representations of the groups $G_i,\ i \in [1, 11]$.

For $G_5 \cong G^(5, 5, 5) \cong L_2(11)$ this is done; there is only one.

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Thank you!