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Constructing group extensions with special properties

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Classification of finite groups

'Die Hauptschwierigkeit besteht dabei nicht in einer Konstruktion aller Gruppen eines bestimmten Typs, sondern in der Angabe eines vollständigen Systems nicht isomorpher Gruppen aus den konstruierten Gruppen.' *(W. Magnus, 1937)*

(Here, the main difficulty is not to construct all groups of a given type, but to provide a complete list of non-isomorphic groups from the constructed groups.)

Methods:

- p-group generation algorithm [Newman, O'Brien]
- Frattini-extension method (solvable groups) [Besche, Eick]
- Upwards extension method [Laue; Betten; Besche, Eick]

Extensions of special type

Let G be a group and A an abelian group.
We say E is an *extension* of A by G if $A \lhd E$ and $G \cong E/A$.

Definition

An extension E of A by G is ...

- ...a *lower central extension* if $\lambda_{c-1}(E) = A$ and $\lambda_c(E) = 1$ for some $c \in \mathbb{N}$.
- ...a *derived series extension* if $E^{(n)} = A$ for some $n \in \mathbb{N}$.

For $\delta \in Z^2(G, A)$ denote by E_δ the corresponding extensions of A by G and write $[\delta] = \delta + B^2(G, A) \in H^2(G, A)$.

For $\gamma, \delta \in Z^2(G, A)$ with $[\gamma] = [\delta]$ we have $E_\gamma \cong E_\delta$.

In general $E_\gamma \cong E_\delta$ does not imply $[\gamma] = [\delta]$.

Strong isomorphism and compatible pairs

Let G act on A via the action homomorphism $G \to \mathrm{Aut}(A)$, $g \mapsto \overline{g}$.

Definition

Let $\gamma, \delta \in Z^2(G, A)$. The extensions E_γ and E_δ are *strongly isomorphic* if there exists an isomorphism from E_γ to E_δ which maps A to A.

Define the group of compatible pairs $\mathrm{Comp}(G, A)$ as

$$\left\{(\kappa, \mu) \in \mathrm{Aut}(G) \times \mathrm{Aut}(A) \mid \mu\overline{g^\kappa} = \overline{g}\mu \text{ for } g \in G\right\}$$

acting on cocycles via $\delta^{(\kappa,\mu)} : G \times G \to A, (g, h) \mapsto (\delta(g^{\kappa^{-1}}, h^{\kappa^{-1}}))^\mu$

Robinson, 1981: E_γ and E_δ are strongly isomorphic if and only if there exists $(\kappa, \mu) \in \mathrm{Comp}(G, A)$ such that $[\gamma^{(\kappa,\mu)}] = [\delta]$.

Identify lower central extensions

Let $\delta \in Z^2(G, A)$ and define

$$\hat{\delta} : \begin{cases} G \times G \to A \\ (g, h) \mapsto \delta(g, h)\delta(h, g)^{-1}\delta((hg)^{-1}, hg)^{-1}\delta((hg)^{-1}, gh). \end{cases}$$

For $H \leqslant Z(G)$ denote with $\hat{\delta}_H$ the restriction of $\hat{\delta}$ on $G \times H$.

Theorem

Suppose that G is nilpotent of class c and acts trivially on A and let $\delta \in Z^2(G, A)$.
Then E_δ is a lower central extension if and only if $\operatorname{im} \hat{\delta}_{\lambda_c(G)} = A$.

Identify derived series extension

Let N denote the last non-trivial term of the derived series of G.

For $\delta \in Z^2(G, A)$ denote with

$$\delta' : \begin{cases} N \times N \to A/[A, N] \\ (g, h) \mapsto \delta(g, h)[A, N] \end{cases}$$

the cocycle in $Z^2(N, A/[A, N])$ induced by δ.

Theorem

Suppose that G is solvable and let N denote the last non-trivial term of the derived series of G. Let A be abelian and $\delta \in Z^2(G, A)$.
Then E_δ is a derived series extension if and only if $\operatorname{im} \widehat{\delta'} = A/[A, N]$.

Implementation

Implementation in GAP for A elementary-abelian.

G is given by a polycyclic presentation with a generating set running through the lower central series respectively the derived series.

The extension E_δ is determined by values (*tails*) in $A \subset E_\delta$ of the relators in the presentation of G. So, δ is represented as a list of tails.

Taking $G \times A$ as underlying set of E_δ, in a central extension we have

$$[(g, a), (h, b)] = ([g, h], \hat{\delta}(g, h)).$$

For $\hat{\delta}_H$ a multiplicative condition holds:

$$\hat{\delta}_H(g_1 g_2, h_1 h_2) = \hat{\delta}_H(g_1, h_1)\hat{\delta}_H(g_1, h_2)\hat{\delta}_H(g_2, h_1)\hat{\delta}_H(g_2, h_2).$$

Implementation cont.

Test im $\hat{\delta}_{\lambda_c(G)} = A$ (the condition for a lower central extension):

- $T \leftarrow$ tails corresponding to $G \times \lambda_c(G)$
- verify whether T forms a basis of A (as vector space)

Test im $\widehat{\delta'} = A/[A, N]$ (the condition for a derived series extension):

- $T \leftarrow$ tails corresponding to commutators from $N \times N$
- verify whether T with $[A, N]$ forms a basis of A (as vector space)

Cases without suitable extensions can be discarded quickly, in particular if $|T| < \dim A - \dim [A, N]$.

Example

A polycyclic presentation for SMALLGROUP(18,4) is given by,

$$G = \langle a, b, c \mid a^2 = b^3 = c^3 = [c, b] = 1, [b, a] = b, [c, a] = c\rangle$$

and acting on $A \cong \mathbb{F}_3^2$ via

$$a \hat{=} \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}, b \hat{=} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} c \hat{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$N = \langle b, c\rangle \cong C_3 \times C_3, \quad [A, N] \cong 1 \times C_3$$

Cocycles are represented by $\mathbb{F}_3$-vectors of length 12 (6 tails $\in A$).

$$|Z^2(G, A)| = 3^6, |B^2(G, A)| = 3^3, |H^2(G, A)| = 3^3.$$

Example cont.

- $\operatorname{im}\widehat{\delta'} = \langle \delta(c,b)[A,N] \rangle \quad (\stackrel{?}{=} A/[A,N])$
- check whether $\mathbb{F}_3^2 = \langle \delta(c,b), (0,1) \rangle \quad (\stackrel{?}{=} A)$
- $|\operatorname{Comp}(G,A)| = 216$

There are 7 orbits of $H^2(G,A)$ under the action of $\operatorname{Comp}(G,A)$.

Two of the orbits contain cosets of cocycles δ with $\delta(c,b) = 0 \in A$.

The remaining five lead to derived series extensions. For example δ with $\delta(c,b) = (1,0)$ and $(0,0)$ otherwise. Then

$$\begin{aligned} E_\delta = \langle a, b, c, d, e \quad | \quad & a^2 = b^3 = c^3 = d^3 = e^3 = 1, \\ & [c,b] = d, [b,a] = b, [c,a] = c, \dots \rangle \end{aligned}$$

All lower central extensions

Remark

Given a group G it is possible to compute all lower central extensions by G because the dimension of the module A is bounded by

$$d\left(G/\lambda_1(G)\right) * d\left(\lambda_c(G)\right).$$

Thank you for your attention!

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