The influence of p-regular class sizes on normal subgroups

María José Felipe

Universidad Politécnica de Valencia (Spain)

Groups St Andrews 2013

in collaboration with Zeinab Akhlaghi and Antonio Beltrán

Conjugacy class sizes

Notation

Let G be a finite group and $x \in G$. We denote by

$$x^G = \{x^g : g \in G\}$$

the conjugacy class of x in G and by

$$\text{cs}(G) = \{|x^G| : x \in G\}.$$

Conjugacy class sizes

Notation

Let G be a finite group and $x \in G$. We denote by

$$x^G = \{x^g : g \in G\}$$

the conjugacy class of x in G and by

$$\mathrm{cs}(G) = \{|x^G| : x \in G\}.$$

It is well-known that there exists a strong relation between $\mathrm{cs}(G)$ and the structure of G.

Conjugacy class sizes

Notation

Let G be a finite group and $x \in G$. We denote by

$$x^G = \{x^g : g \in G\}$$

the conjugacy class of x in G and by

$$\mathrm{cs}(G) = \{|x^G| : x \in G\}.$$

It is well-known that there exists a strong relation between $\mathrm{cs}(G)$ and the structure of G.

Theorem (N. Itô, 1953)

If $|\mathrm{cs}(G)| = 2$, then $G = P \times A$ with P a p-subgroup, for some prime p, and $A \subseteq \mathbf{Z}(G)$.

Conjugacy class sizes of p-regular elements

Notation

Let p be a prime number and G be a finite group.

An element $x \in G$ is said to be a p-**regular element** (or a p'-element) if the order $o(x)$ is not divisible by p.

We can consider the set

$$\mathrm{cs}_{p'}(G) = \{|x^G| : x \text{ is a } p\text{-regular element of } G\}.$$

Conjugacy class sizes of p-regular elements

Notation

Let p be a prime number and G be a finite group.

An element $x \in G$ is said to be a p-**regular element** (or a p'-element) if the order $o(x)$ is not divisible by p.

We can consider the set

$$\mathrm{cs}_{p'}(G) = \{|x^G| : x \text{ is a } p\text{-regular element of } G\}.$$

Some questions:

- What can be said about the structure of G from $\mathrm{cs}_{p'}(G)$?

Conjugacy class sizes of p-regular elements

Notation

Let p be a prime number and G be a finite group.

An element $x \in G$ is said to be a p-**regular element** (or a p'-element) if the order $o(x)$ is not divisible by p.

We can consider the set

$$\mathrm{cs}_{p'}(G) = \{|x^G| : x \text{ is a } p\text{-regular element of } G\}.$$

Some questions:

- What can be said about the structure of G from $\mathrm{cs}_{p'}(G)$?
- If H is a p-complement of G, which is the relation between $\mathrm{cs}_{p'}(G)$ and $\mathrm{cs}(H)$?

Lemma

Let H be a p-complement of a finite group G. Let $x \in H$.
(i) Then $|x^G|_{p'}$ divides $|x^H|$.

Lemma

Let H be a p-complement of a finite group G. Let $x \in H$.
(i) Then $|x^G|_{p'}$ divides $|x^H|$.
(ii) If $|x^G|$ is a p'-number, then $|x^H| = |x^G|$.

Lemma

Let H be a p-complement of a finite group G. Let $x \in H$.
(i) Then $|x^G|_{p'}$ divides $|x^H|$.
(ii) If $|x^G|$ is a p'-number, then $|x^H| = |x^G|$.
(iii) If $H \trianglelefteq G$, then $|x^H| = |x^G|_{p'}$.

Lemma

Let H be a p-complement of a finite group G. Let $x \in H$.
(i) Then $|x^G|_{p'}$ divides $|x^H|$.
(ii) If $|x^G|$ is a p'-number, then $|x^H| = |x^G|$.
(iii) If $H \trianglelefteq G$, then $|x^H| = |x^G|_{p'}$.

Question:

- In general, is $|x^H|$ a divisor of $|x^G|$?

Lemma

Let H be a p-complement of a finite group G. Let $x \in H$.
(i) Then $|x^G|_{p'}$ divides $|x^H|$.
(ii) If $|x^G|$ is a p'-number, then $|x^H| = |x^G|$.
(iii) If $H \trianglelefteq G$, then $|x^H| = |x^G|_{p'}$.

Question:

- In general, is $|x^H|$ a divisor of $|x^G|$?

This question is false.

Lemma

Let H be a p-complement of a finite group G. Let $x \in H$.
(i) Then $|x^G|_{p'}$ divides $|x^H|$.
(ii) If $|x^G|$ is a p'-number, then $|x^H| = |x^G|$.
(iii) If $H \trianglelefteq G$, then $|x^H| = |x^G|_{p'}$.

Question:

- In general, is $|x^H|$ a divisor of $|x^G|$?

This question is false.

Example:

The symmetric group $H = S_4$ is a Hall $\{2,3\}$-subgroup of the symmetric group $G = S_5$ (that is, a 5-complement of S_5). The sets of class sizes are

$$\text{cs}(H) = \{1,3,6,8\} \text{ and } \text{cs}_{p'}(G) = \{1,10,15,20,30\}.$$

Let $x = (1,2,3) \in H$, the class size $|x^H| = 8$ and to $|x^G| = 20$.

Question:

- In general, is $|\mathrm{cs}(H)| \leq |\mathrm{cs}_{p'}(G)|$?

Question:

- In general, is $|\text{cs}(H)| \leq |\text{cs}_{p'}(G)|$?

This question neither is true.

Question:

- In general, is $|\mathrm{cs}(H)| \leq |\mathrm{cs}_{p'}(G)|$?

This question neither is true.

Example:

The quaternion group Q_8 acts on $T = [\mathbb{Z}_5 \times \mathbb{Z}_5]\mathbb{Z}_3$.

We consider $G = [T]Q_8$ (SmallGroup(600, 57) in GAP).

Let $H = [\mathbb{Z}_5 \times \mathbb{Z}_5]Q_8$ be a 3-complement of G. We have

$$\mathrm{cs}(H) = \{1, 2, 4, 10, 50\};$$

$$\mathrm{cs}(G) = \mathrm{cs}_{p'}(G) = \{1, 6, 30, 50\}.$$

Question:

- In general, is $|\mathrm{cs}(H)| \leq |\mathrm{cs}_{p'}(G)|$?

This question neither is true.

Example:

The quaternion group Q_8 acts on $T = [\mathbb{Z}_5 \times \mathbb{Z}_5]\mathbb{Z}_3$.

We consider $G = [T]Q_8$ (SmallGroup(600, 57) in GAP).

Let $H = [\mathbb{Z}_5 \times \mathbb{Z}_5]Q_8$ be a 3-complement of G. We have

$$\mathrm{cs}(H) = \{1, 2, 4, 10, 50\};$$

$$\mathrm{cs}(G) = \mathrm{cs}_{p'}(G) = \{1, 6, 30, 50\}.$$

Therefore, there is not relation between $|\mathrm{cs}(H)|$ and $|\mathrm{cs}_{p'}(G)|$.

Structure of p-complements and p-regular class sizes

New topic:

Some recent results have indicated that the structure of G and its p-complements are closely related to the set $\mathrm{cs}_{p'}(G)$.

Structure of p-complements and p-regular class sizes

New topic:

Some recent results have indicated that the structure of G and its p-complements are closely related to the set $\mathrm{cs}_{p'}(G)$.

But studying this relation seems a difficult problem, even when G is a p-solvable group.

Structure of p-complements and p-regular class sizes

New topic:

Some recent results have indicated that the structure of G and its p-complements are closely related to the set $\mathrm{cs}_{p'}(G)$.

But studying this relation seems a difficult problem, even when G is a p-solvable group.

Theorem (A. Camina,1974)

If $|\mathrm{cs}_{p'}(G)| = 2$,*then* G *is solvable.*

New topic:

Some recent results have indicated that the structure of G and its p-complements are closely related to the set $\mathrm{cs}_{p'}(G)$.

But studying this relation seems a difficult problem, even when G is a p-solvable group.

Theorem (A. Camina,1974)

If $|\mathrm{cs}_{p'}(G)| = 2$, then G is solvable.

Theorem (E.Alemany-A.Beltrán-M.J.Felipe, 2009)

Let H be a p-complement of G. If $|\mathrm{cs}_{p'}(G)| = 2$, then either H is abelian or $H = Q \times A$ with $Q \in Syl_q(G)$ for $q \neq p$ and then $G = PQ \times A$, with $P \in Syl_p(G)$ and $A \subseteq \mathbf{Z}(G)$.

Structure of normal subgroups and G-classes

Structure of normal subgroups and G-classes

Notation

Let N be a normal subgroup of G. Then

$$N = \cup_{x \in N} x^G$$

Let $x \in N$. The class x^G is called a **G-class of N**. We denote by

$$\mathrm{cs}_G(N) = \{|x^G| : x \in N\} \subseteq \mathrm{cs}(G).$$

Notation

Let N be a normal subgroup of G. Then

$$N = \cup_{x \in N} x^G$$

Let $x \in N$. The class x^G is called a **G-class of N**. We denote by

$$\mathrm{cs}_G(N) = \{|x^G| : x \in N\} \subseteq \mathrm{cs}(G).$$

It can be seen that there is no relation between the cardinalities $|\mathrm{cs}_G(N)|$ and $|\mathrm{cs}(N)|$.

Structure of normal subgroups and G-classes

Notation

Let N be a normal subgroup of G. Then

$$N = \cup_{x \in N} x^G$$

Let $x \in N$. The class x^G is called a **G-class of N**. We denote by

$$\mathrm{cs}_G(N) = \{|x^G| : x \in N\} \subseteq \mathrm{cs}(G).$$

It can be seen that there is no relation between the cardinalities $|\mathrm{cs}_G(N)|$ and $|\mathrm{cs}(N)|$.

Another new topic:

Recent results have put forward that there exists a strong relation between $\mathrm{cs}_G(N)$ and the structure of N.

The influence of p-regular class sizes on normal subgroups

Theorem 1 (Akhlaghi-Beltrán-Felipe-Khatami, 2011):

Let G be a **finite p-solvable group** *and N be a normal subgroup of G. Suppose that N has two p-regular G-class sizes for some prime p. Then N has nilpotent p-complements.*

Theorem 1 (Akhlaghi-Beltrán-Felipe-Khatami, 2011):

Let G be a **finite p-solvable group** *and N be a normal subgroup of G. Suppose that N has two p-regular G-class sizes for some prime p. Then N has nilpotent p-complements.*

Question:

- Can the hypothesis of p-solvability of G be eliminated?

The influence of p-regular class sizes on normal subgroups

Theorem 1 (Akhlaghi-Beltrán-Felipe-Khatami, 2011):

Let G be a **finite p-solvable group** *and N be a normal subgroup of G. Suppose that N has two p-regular G-class sizes for some prime p. Then N has nilpotent p-complements.*

Question:

- Can the hypothesis of p-solvability of G be eliminated?

Theorem 2 (Akhlaghi-Beltrán-Felipe, 2013):

If N is a **solvable** *normal subgroup of a group G with two G-class sizes of p-regular elements, then N has nilpotent p-complements.*

Theorem 1 (Akhlaghi-Beltrán-Felipe-Khatami, 2011):

Let G be a **finite p-solvable group** *and N be a normal subgroup of G. Suppose that N has two p-regular G-class sizes for some prime p. Then N has nilpotent p-complements.*

Question:

- Can the hypothesis of p-solvability of G be eliminated?

Theorem 2 (Akhlaghi-Beltrán-Felipe, 2013):

If N is a **solvable** *normal subgroup of a group G with two G-class sizes of p-regular elements, then N has nilpotent p-complements.*

Question:

- We wonder if a normal subgroup of G having two p-regular G-class sizes is solvable.

The influence of p-regular class sizes on normal subgroups

Theorem A (Akhlaghi-Beltrán-Felipe, 2013):

If N is a normal subgroup of G with two G-class sizes of p-regular elements, then N is solvable.

The influence of p-regular class sizes on normal subgroups

Theorem A (Akhlaghi-Beltrán-Felipe, 2013):

If N is a normal subgroup of G with two G-class sizes of p-regular elements, then N is solvable.

Corollary B (Akhlaghi-Beltrán-Felipe, 2013):

If N is a normal subgroup of a group G with two G-class sizes of p-regular elements, then either N has abelian p-complements or $N = RP \times A$, where P is a Sylow p-subgroup, R is a Sylow r-subgroup for $r \neq p$ and $A \subseteq \mathbf{Z}(G)$.

Theorem A (Akhlaghi-Beltrán-Felipe, 2013):

If N is a normal subgroup of G with two G-class sizes of p-regular elements, then N is solvable.

Corollary B (Akhlaghi-Beltrán-Felipe, 2013):

If N is a normal subgroup of a group G with two G-class sizes of p-regular elements, then either N has abelian p-complements or $N = RP \times A$, where P is a Sylow p-subgroup, R is a Sylow r-subgroup for $r \neq p$ and $A \subseteq \mathbf{Z}(G)$.

Corollary C (Alemany-Beltrán-Felipe, 2011):

Let N be a normal subgroup of a finite group G such that $|cs_G(N)| = 2$. Then either N is abelian or $N = R \times A$, where R is a Sylow r-subgroup of N for some prime r and A is central in G.

Preliminary results of the proof of Theorem A

The prime graph

Let G be a finite group. The **prime graph** $\Gamma(G)$ *of G is defined as follows. The vertices of* $\Gamma(G)$ *are the primes dividing the order of G and two distinct vertices r and s are joined by an edge if there is an element in G of order rs.*

Preliminary results of the proof of Theorem A

The prime graph

Let G be a finite group. The **prime graph** $\Gamma(G)$ *of G is defined as follows. The vertices of $\Gamma(G)$ are the primes dividing the order of G and two distinct vertices r and s are joined by an edge if there is an element in G of order rs.*

The prime graph $\Gamma(G)$ is a **tree** *if any two primes are connected by exactly one simple path and it is a* **forest** *if it is a disjoint union of trees.*

Theorem 3 (M.S. Lucido, 2002)

Let G be a finite non-abelian simple group. If $\Gamma(G)$ is a forest then G is one of the following simple groups: A_5, A_6, A_7, A_8, M_{11}, M_{22}, $PSL_4(3)$, $B_2(3)$, $G_2(3)$, $U_4(3)$, $U_5(2)$, ${}^2F_4(2)'$, or belongs to one of the families: $PSL_2(q)$, $PSL_3(q)$, $PSU_3(q)$, $Sz(q^2)$ with $q^2 = 2^f$ or $q = 2^{f^2}$ with f an odd prime, and $Ree(3^f)$, with f an odd prime.

Preliminary results of the proof of Theorem A

Theorem 4 (Akhlaghi-Beltrán-Felipe-Khatami, 2011):

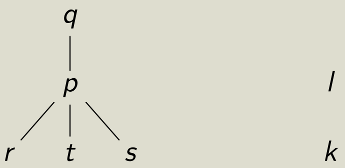
Let N be a normal subgroup of a group G which has two p-regular G-class sizes for some prime p. Then either N has abelian p-complements or all p-regular elements of $N/(N \cap \mathbf{Z}(G))$ have prime power order.

Preliminary results of the proof of Theorem A

Theorem 4 (Akhlaghi-Beltrán-Felipe-Khatami, 2011):

Let N be a normal subgroup of a group G which has two p-regular G-class sizes for some prime p. Then either N has abelian p-complements or all p-regular elements of $N/(N \cap \mathbf{Z}(G))$ have prime power order.

As a consequence, either N has abelian p-complements or the prime graph $\Gamma(N/(N \cap \mathbf{Z}(G)))$ is a forest:



Sketch of the proof of Theorem A.

(1) We argue by minimal counterexample.

Sketch of the proof of Theorem A.

(1) We argue by minimal counterexample.
(2) By Theorem 4, we assume that $N/(N \cap \mathbf{Z}(G))$ does not have any p-regular element whose order is divisible by two primes and the prime graph $\Gamma(N/(N \cap \mathbf{Z}(G)))$ is a forest.

Sketch of the proof of Theorem A.

(1) We argue by minimal counterexample.
(2) By Theorem 4, we assume that $N/(N \cap \mathbf{Z}(G))$ does not have any p-regular element whose order is divisible by two primes and the prime graph $\Gamma(N/(N \cap \mathbf{Z}(G)))$ is a forest.
(3) It is easy to prove that $|N/(N \cap \mathbf{Z}(G))|_{p'}$ divides $|N \cap \mathbf{Z}(G)|$ by using the class equation.

Sketch of the proof of Theorem A.

(1) We argue by minimal counterexample.
(2) By Theorem 4, we assume that $N/(N \cap \mathbf{Z}(G))$ does not have any p-regular element whose order is divisible by two primes and the prime graph $\Gamma(N/(N \cap \mathbf{Z}(G)))$ is a forest.
(3) It is easy to prove that $|N/(N \cap \mathbf{Z}(G))|_{p'}$ divides $|N \cap \mathbf{Z}(G)|$ by using the class equation.
(4) Let N/K be a chief factor of G such that $N \cap \mathbf{Z}(G) \subseteq K$.

Sketch of the proof of Theorem A.

(1) We argue by minimal counterexample.
(2) By Theorem 4, we assume that $N/(N \cap \mathbf{Z}(G))$ does not have any p-regular element whose order is divisible by two primes and the prime graph $\Gamma(N/(N \cap \mathbf{Z}(G)))$ is a forest.
(3) It is easy to prove that $|N/(N \cap \mathbf{Z}(G))|_{p'}$ divides $|N \cap \mathbf{Z}(G)|$ by using the class equation.
(4) Let N/K be a chief factor of G such that $N \cap \mathbf{Z}(G) \subseteq K$. By minimality of N, we obtain that K is a solvable subgroup.

Sketch of the proof of Theorem A.

(1) We argue by minimal counterexample.
(2) By Theorem 4, we assume that $N/(N \cap \mathbf{Z}(G))$ does not have any p-regular element whose order is divisible by two primes and the prime graph $\Gamma(N/(N \cap \mathbf{Z}(G)))$ is a forest.
(3) It is easy to prove that $|N/(N \cap \mathbf{Z}(G))|_{p'}$ divides $|N \cap \mathbf{Z}(G)|$ by using the class equation.
(4) Let N/K be a chief factor of G such that $N \cap \mathbf{Z}(G) \subseteq K$.
By minimality of N, we obtain that K is a solvable subgroup.
By Theorem 2, we have K has nilpotent p-complements and, as a consequence, the order of $K/(N \cap \mathbf{Z}(G))$ is divisible by at most two primes $\{p, r\}$.

Sketch of the proof of Theorem A.

(1) We argue by minimal counterexample.
(2) By Theorem 4, we assume that $N/(N \cap \mathbf{Z}(G))$ does not have any p-regular element whose order is divisible by two primes and the prime graph $\Gamma(N/(N \cap \mathbf{Z}(G)))$ is a forest.
(3) It is easy to prove that $|N/(N \cap \mathbf{Z}(G))|_{p'}$ divides $|N \cap \mathbf{Z}(G)|$ by using the class equation.
(4) Let N/K be a chief factor of G such that $N \cap \mathbf{Z}(G) \subseteq K$.
By minimality of N, we obtain that K is a solvable subgroup.
By Theorem 2, we have K has nilpotent p-complements and, as a consequence, the order of $K/(N \cap \mathbf{Z}(G))$ is divisible by at most two primes $\{p, r\}$.
(5) By (2), the chief factor N/K does not have any p-regular element whose order is divisible by two primes and necessarily $N/K \cong S$, with S a simple group whose prime graph is a forest.

Sketch of the proof of Theorem A.

(6) It is not difficult to show

$$(N \cap \mathbf{Z}(G))_{\{r,p\}'} \cong K/\mathbf{O}_{\{r,p\}}(N) = \mathbf{Z}(N/\mathbf{O}_{\{r,p\}}(N));$$

$$\frac{N/\mathbf{O}_{\{r,p\}}(N)}{\mathbf{Z}(N/\mathbf{O}_{\{r,p\}}(N))} \cong N/K \cong S.$$

(6) It is not difficult to show

$$(N \cap \mathbf{Z}(G))_{\{r,p\}'} \cong K/\mathbf{O}_{\{r,p\}}(N) = \mathbf{Z}(N/\mathbf{O}_{\{r,p\}}(N));$$

$$\frac{N/\mathbf{O}_{\{r,p\}}(N)}{\mathbf{Z}(N/\mathbf{O}_{\{r,p\}}(N))} \cong N/K \cong S.$$

Hence, $N/\mathbf{O}_{\{r,p\}}(N)$ is a quasi-simple group and $|N \cap \mathbf{Z}(G)|_{\{r,p\}'}$ divides $|M(S)|$, where $M(S)$ is the Schur multiplier of S.

(6) It is not difficult to show

$$(N \cap \mathbf{Z}(G))_{\{r,p\}'} \cong K/\mathbf{O}_{\{r,p\}}(N) = \mathbf{Z}(N/\mathbf{O}_{\{r,p\}}(N));$$

$$\frac{N/\mathbf{O}_{\{r,p\}}(N)}{\mathbf{Z}(N/\mathbf{O}_{\{r,p\}}(N))} \cong N/K \cong S.$$

Hence, $N/\mathbf{O}_{\{r,p\}}(N)$ is a quasi-simple group and $|N \cap \mathbf{Z}(G))|_{\{r,p\}'}$ divides $|M(S)|$, where $M(S)$ is the Schur multiplier of S.
(7) By (3), we have $|N/\mathbf{O}_{\{r,p\}}(N)|_{\{r,p\}'}$ divides $|M(S)|$.

Sketch of the proof of Theorem A.

(6) It is not difficult to show

$$(N \cap \mathbf{Z}(G))_{\{r,p\}'} \cong K/\mathbf{O}_{\{r,p\}}(N) = \mathbf{Z}(N/\mathbf{O}_{\{r,p\}}(N));$$

$$\frac{N/\mathbf{O}_{\{r,p\}}(N)}{\mathbf{Z}(N/\mathbf{O}_{\{r,p\}}(N))} \cong N/K \cong S.$$

Hence, $N/\mathbf{O}_{\{r,p\}}(N)$ is a quasi-simple group and $|N \cap \mathbf{Z}(G))|_{\{r,p\}'}$ divides $|M(S)|$, where $M(S)$ is the Schur multiplier of S.
(7) By (3), we have $|N/\mathbf{O}_{\{r,p\}}(N)|_{\{r,p\}'}$ divides $|M(S)|$.
(8) Therefore, we obtain that $|S|$ divides $|M(S)|p^{\alpha}q^{\beta}$, for some α and β.

Sketch of the proof of Theorem A.

(6) It is not difficult to show

$$(N \cap \mathbf{Z}(G))_{\{r,p\}'} \cong K/\mathbf{O}_{\{r,p\}}(N) = \mathbf{Z}(N/\mathbf{O}_{\{r,p\}}(N));$$

$$\frac{N/\mathbf{O}_{\{r,p\}}(N)}{\mathbf{Z}(N/\mathbf{O}_{\{r,p\}}(N))} \cong N/K \cong S.$$

Hence, $N/\mathbf{O}_{\{r,p\}}(N)$ is a quasi-simple group and $|N \cap \mathbf{Z}(G))|_{\{r,p\}'}$ divides $|M(S)|$, where $M(S)$ is the Schur multiplier of S.
(7) By (3), we have $|N/\mathbf{O}_{\{r,p\}}(N)|_{\{r,p\}'}$ divides $|M(S)|$.
(8) Therefore, we obtain that $|S|$ divides $|M(S)|p^{\alpha}q^{\beta}$, for some α and β.
(9) Finally, we can check that this property is not possible for the simple groups listed by M.S. Lucido.

Sketch of the proof of Theorem A.

(6) It is not difficult to show

$$(N \cap \mathbf{Z}(G))_{\{r,p\}'} \cong K/\mathbf{O}_{\{r,p\}}(N) = \mathbf{Z}(N/\mathbf{O}_{\{r,p\}}(N));$$

$$\frac{N/\mathbf{O}_{\{r,p\}}(N)}{\mathbf{Z}(N/\mathbf{O}_{\{r,p\}}(N))} \cong N/K \cong S.$$

Hence, $N/\mathbf{O}_{\{r,p\}}(N)$ is a quasi-simple group and $|N \cap \mathbf{Z}(G))|_{\{r,p\}'}$ divides $|M(S)|$, where $M(S)$ is the Schur multiplier of S.
(7) By (3), we have $|N/\mathbf{O}_{\{r,p\}}(N)|_{\{r,p\}'}$ divides $|M(S)|$.
(8) Therefore, we obtain that $|S|$ divides $|M(S)|p^{\alpha}q^{\beta}$, for some α and β.
(9) Finally, we can check that this property is not possible for the simple groups listed by M.S. Lucido. The most complicated cases are $PSL_2(q)$, $PSL_3(q)$, $PSU_3(q)$. □

References:

- Z. Akhlaghi, A. Beltrán A and M.J. Felipe, The influence of p-regular class sizes on normal subgroups, to appear J. Group Theory.
- Z. Akhlaghi, A. Beltrán, M.J. Felipe, M. Khatami, Normal subgroups and p-regular G-class sizes. J. Algebra **336** (2011), 236-241.
- E. Alemany, A. Beltrán and M.J. Felipe, Nilpotency of normal subgroups having two G-class sizes. Proc. Amer. Math. Soc. **139** (2011), 2663-2669.

Thank you very much

for your attention!