Turner's Theorem is not First Order

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August 2013

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We say a group is a **Turner group** just in case nonprojectibility implies being a test element. We call a group a **vacuous Turner group** provided every element of the group lies in a proper retract of the group.
We also say a group is **curious** if it is elementarily equivalent to a Turner group.

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Since finite gp.'s are stably hyperbolic, O&T recovers Kearnes' result.

Logical Preliminaries

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Two groups G and H are elementarily equivalent provided that they satisfy precisely the same sentences of L_0, i.e., $\mathrm{Th}(G) = \mathrm{Th}(H)$.
Any group which is elementarily equivalent to a nonabelian free group is called **elementarily free.**
Examples of groups which are elementarily free but not free are surface groups both orientable and non-orientable of sufficiently high genus. (For oreintable of genus ≥ 2 and non-orientable of genus ≥ 4)

Logical Preliminaries Cont'd.

A group which has the same universal theory as any nonabelian free group, i.e., satisfies the same universal sentences
$(\forall \overline{x}(\varphi(\overline{x})$ where $\overline{x}$ is a tuple of variables of L_0 and φ is a formula of L_0 which contains at most the variables in $\overline{x}$ but contains no quantifiers.)
as any nonabelian free group is said to be **universally free.**
Of course, being elementarily free $\Longrightarrow$ being universally free

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A group G is **fully residually free** if given any finite set $S \subseteq G \backslash \{1\}$ $\exists$ a homomrphism $\varphi : G \longrightarrow F$ where F is a free group and $\varphi(s) \neq 1$ for all $s \in S$.

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(Remeslennikov) Let G be a f.g. universally free group. Then G is residually free.

Group Theoretical Preliminaries Cont'd.

Theorem

(Re & GS) Suppose the group G is nonabelian and residually free. Then the following are equivalent:
(1) G is fully residually free
(2) G is CT
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(Kh & M): Let G be a f.g. group. If G is fully residually free and has cyclic centralizers of nontrivial elements, then G is hyperbolic.

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Let x be a variable and $\Phi(x)$ be a set of formulas, $\varphi(x)$, of L_0. Here the formulas $\varphi(x)$ can contain one or more free occurrences of at most the variable x -but no other variable.

If G is a gp. and $g \in G$ then we say $\Phi(g)$ holds provided $\varphi(g)$ holds in G for every formula $\varphi(x) \in \Phi(x)$.

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(Nondefinability Theorem.) (1) There is no set $N(x)$ *of formulas of* L_0 *s.t. for an arbitrary gp.* G *and an arbitrary element* $g \in G$, $N(g)$ *holds iff* g *is nonprojectible.*
(2) There is no set $T(x)$ *of formulas of* L_0 *s.t. for an arbitrary gp.* G *and an arbitrary element* $g \in G$, $T(g)$ *holds iff* g *is a test element.*

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(3) Are infinitley generated elementarily free gp.'s Turner gp's ?