On the relation lifting and relation gap problem

Jens Harlander, Boise State University

August 2013

The Basic Problem

Let $F = F(x_1, ..., x_n)$ be a free group and $G = F/N$ a finitely presented group. How can we determine a minimal set of normal generators of N?

Example: the group $(\mathbb{Z}_2 \times \mathbb{Z}) * (\mathbb{Z}_3 \times \mathbb{Z})$ can be presented as

$$\langle a, b, c, d \mid a^2 = 1, ab = ba, c^3 = 1, cd = dc \rangle.$$

Can we get away with just three relations?

Relation Modules

The conjugation action of F on N provides a $\mathbb{Z}G$-module structure on $N_{ab} = N/[N,N]$. This module is the relation module of the presentation F/N of G. We have

$$d_F(N) \geq d_G(N/[N,N]) \geq d(N/[F,N])$$

and $d(N/[F,N]) = d(F) - tfr(H_1(G)) + d(H_2(G))$. Here

- $d_F(-)$ denotes the minimal number of F-group generators,
- $d_G(-)$ denotes the minimal number of G-module generators,
- $d(-)$ denotes the minimal number of generators, and
- $tfr(-)$ denotes the torsion free rank.

Examples

Consider the presentation

$$F/N = \langle a, b \mid a^2 = 1, ab = ba \rangle$$

of $G = \mathbb{Z}_2 \times \mathbb{Z}$. Note that $d(N/[F, N]) = 2 - 1 + 1 = 2$. Thus we have indeed displayed a minimal set of relations.
Consider the presentation

$$F/N = \langle a, b, c, d \mid a^2 = 1, ab = ba, c^3 = 1, cd = dc \rangle$$

of $G = (\mathbb{Z}_2 \times \mathbb{Z}) * (\mathbb{Z}_3 \times \mathbb{Z})$. Note that $d(N/[F, N]) = 4 - 3 + 2 = 3$. Are we displaying a minimal set of relations?

Three Questions

Given a finite presentation of a group G.

- **Relation gap question**: can there be a relation gap $d_F(N) - d_G(N_{ab}) > 0$?
- **Relation lifting question**: can generators of N_{ab} be lifted to provide a complete set of relations?
- **Geometric realization question**: is a given algebraic 2-complex

$$\mathbb{Z}G^m \to \mathbb{Z}G^n \to \mathbb{Z}G^k \to \mathbb{Z} \to 0$$

  chain homotopy equivalent to one arising as the chain complex of the universal covering of a 2-complex with fundamental group G?

Relation Gap

Bestvina and Brady 1997 constructed a presentation F/N of a finitely generated group G with infinite relation gap: N_{ab} is finitely generated but the group G does not admit a finite presentation.

Positive relation gaps are not known for finite presentations F/N. Examples where a positive relation gap seems likely will be presented later.

Relation Lifting

This question arose in work of C. T. C. Wall (1965). M. Dunwoody (1972) provided an example where lifting is not possible.

Consider the presentation $F/N = \langle a, b \mid a^5 = 1 \rangle$ of the group $G = \mathbb{Z}_5 * \mathbb{Z}$. Note that $(1 - a + a^2)(a + a^2 - a^4) = 1$, so $1 - a + a^2$ is a non-trivial unit in $\mathbb{Z}G$. Hence so is $(1 - a + a^2)b$. Hence $(1 - a + a^2)b \cdot a^5[N, N] = (ba^5b^{-1})(aba^5b^{-1}a^{-1})(a^2ba^5b^{-1}a^{-2})[N, N]$ is a generator for the relation module. This generator can not be lifted. If $\langle\langle a^5 \rangle\rangle = \langle\langle r \rangle\rangle$, then $r = wa^{\pm 5}w^{-1}$ by 1-relator group theory. one can show that $r[N, N] = w \cdot a^{\pm 5} \neq (1 - a + a^2)b \cdot a^5[N, N]$.

Geometric Realization

Since the relation module of a presentation $G = F(x_1, ..., x_n)/N$ is isomorphic to $H_1(\Gamma)$, where Γ is the Cayley graph of G associated with the generating set $\{x_1, ..., x_n\}$, a choice of relation module generators gives rise to an partial resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$

$$\mathbb{Z}G^m \xrightarrow{\partial_2} \mathbb{Z}G^n \xrightarrow{\partial_1} \mathbb{Z}G^k \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

We call such a partial resolution an algebraic 2-complex for G. We say an algebraic 2-complex for G is geometrically realizable if it is chain homotopy equivalent to a partial resolution that arises as the augmented chain complex of the universal covering of a 2-complex with fundamental group G.

Example 1, Epstein (1961)

Consider the presentation

$$F/N = \langle a, b, c, d \mid [a, b] = 1, a^m = 1, [c, d] = 1, c^n = 1 \rangle$$

of the group $G = (\mathbb{Z}_m \times \mathbb{Z}) * (\mathbb{Z}_n \times \mathbb{Z})$, where m and n are relatively prime. Note that $d(N/[F, N) = 4 - 3 + 2 = 3$. D. Epstein asked (1960) if this presentation is efficient, i.e. if $d_F(N) = d(N/[F, N])$. K. Gruenberg and P. Linnell (2008) showed that $d_G(N/[N, N]) = 3$.

Example 2, Bridson/Tweedale (2007)

These examples are in the spirit of the Epstein example, but an unexpectedly small set of relation module generators can be seen more easily. Let Q_n be the group defined by

$$\langle a, b, x \mid a^n = 1, b^n = 1, [a, b] = 1, xax^{-1} = b\rangle.$$

Note that

$$\langle a, x \mid a^n = 1, [a, xax^{-1}] = 1\rangle$$

also presents Q_n. Then

$$F/N = \langle a, x, b, y \mid a^m = 1, [a, xax^{-1}] = 1, b^n = 1, [b, yby^{-1}] = 1\rangle$$

is a presentation of the free product $Q_m * Q_n$.

Example 2, continued

$Q_m * Q_n$ is presented by

$$F/N = \langle a, x, b, y \mid a^m = 1, [a, xax^{-1}] = 1, b^n = 1, [b, yby^{-1}] = 1\rangle.$$

Let $q_m = (m+1)^m - 1$ and $q_n = (n+1)^n - 1$. Bridson and Tweedale show that in case q_m and q_n are relatively prime the elements

- $\rho_m(a, x) = [xax^{-1}, a]a^{-m}[N, N]$
- $\rho_n(b, y) = [yby^{-1}, b]b^{-n}[N, N]$
- $a^m b^{-n}[N, N]$

generate the relation module in case q_m and q_n are relatively prime.

Example 3, (1998)

Let F_1/N_1 and F_2/N_2 be finite presentations of groups G_1 and G_2, respectively. Let H be a finitely generated subgroup of both G_1 and G_2 and let F/N be the standard presentation of the amalgamated product $G = G_1 *_H G_2$. One can show that

$$d_G(N_{ab}) \leq d_{G_1}(N_{1\,ab}) + d_{G_2}(N_{2\,ab}) + d_H(IH),$$

where IH is the augmentation ideal of H. Cossey, Gruenberg, and Kovacs (1974) showed that $d_{H^n}(IH^n) = d_H(IH)$ in case H is a finite perfect group. Since $d(H^n) \to \infty$ as $n \to \infty$ one can produce arbitrarily large generation gaps $d(H^n) - d_{H^n}(IH^n)$. This leads to unexpectedly small generating sets for the relation module N_{ab} for for presentations F/N of $G_1 *_{H^n} G_1$. Shifts the generation gap into (hopefully) a relation gap.

Stabilization

Metzler and Hog-Angeloni (1990) studied stabilization $K \vee S ... \vee S$ using the 2-complex $S = \langle a, b \mid [a, b], a^2, b^4 \rangle$. Using the 2-sphere for stabilization is standard, but for Metzler and Hog-Angeloni it was important to stabilize with a 2-complex at the minimal Euler characteristic level.

Theorem, (1996) Given a finite presentation F/N. Then there exists $k \geq 0$ such that

$$F/N * \ k \text{ free factors } \langle a, b \mid a^2 = 1, b^2 = 1, [a, b] = 1 \rangle$$

does not have a relation gap.

1-relator groups

Let $F/N = \langle x_1, ..., x_n \mid r \rangle$ be a presentation of a torsion-free one-relator group G. Then the relation module N_{ab} is isomorphic to $\mathbb{Z}G$. Let α and β be left module generators of $\mathbb{Z}G$. Let $\partial_{\alpha,\beta} : \mathbb{Z}G \oplus \mathbb{Z}G \to N_{ab}$ be the homomorphism that sends $e_1 = (1,0)$ to $\alpha \cdot r[N,N]$ and $e_2 = (0,1)$ to $\beta \cdot r[N,N]$. Since the relation module N_{ab} is isomorphic to the kernel of $\partial_1 : \mathbb{Z}G^n \to \mathbb{Z}G$ that sends the basis element e_i to $x_i - 1$, $i = 1, ..., n$, we obtain an algebraic 2-complex $\mathcal{K}_{\alpha,\beta}$

$$\mathbb{Z}G \oplus \mathbb{Z}G \stackrel{\partial_{\alpha,\beta}}{\to} \mathbb{Z}G \oplus \mathbb{Z}G \stackrel{\partial_1}{\to} \mathbb{Z}G \stackrel{\epsilon}{\to} \mathbb{Z} \to 0.$$

This construction provides easy access to examples relevant for relation lifting and geometric realization.

Another example of Dunwoody (1976)

Let G be the trefoil group presented by $F/N = \langle a, b \mid a^2 = b^3 \rangle$. Then

- $\alpha = 1 + a + a^2$
- $\beta = 1 + b + b^2 + b^3$

generate the left module $\mathbb{Z}G$. One obtains an algebraic 2-complex $\mathcal{K}_{\alpha,\beta}$. Dunwoody showed that $H_2(\mathcal{K}_{\alpha,\beta})$ is stably-free but not free. In particular $\mathcal{K}_{\alpha,\beta}$ is not chain homotopically equivalent to the chain complex of the universal covering of $\langle a, b \mid a^2 b^{-3} \rangle \vee S^2$.

Dunwoody's example, continued

Is the algebraic 2-complex $\mathcal{K}_{\alpha,\beta}$ geometrically realizable? Yes. In fact Dunwoody shows that the relation module generator lift and

$$\langle a, b \mid (r)(ara^{-1})(a^2ra^{-2}) = 1, (r)(brb^{-1})(b^2rb^{-2})(b^3rb^{-3}) = 1\rangle$$

is indeed a presentation of G. This provided the first example of different homotopy types of 2-complexes K and L with the same fundamental group G and Euler characteristic $\chi(K) = \chi(L) = \chi_{min}(G) + 1$.

Homotopy classification

The homotopy classification of 2-complexes with fundamental group G, where $G = F(x_1, ..., x_n)$, or $G = \mathbb{Z} \times \mathbb{Z}$ is complete. In the first case

$$(S^1 \vee ... \vee S^1) \vee S^2 \vee ... \vee S^2$$

(n copies of S^1) is a complete list, and in the second case

$$(S^1 \times S^1) \vee S^2 ... \vee S^2$$

is a complete list. Also, the homotopy classification of algebraic 2-complexes coincides with the homotopy classification of 2-complexes.

Homotopy classification, continued

This follows from MacLane-Whitehead (1950): the homotopy type of a 2-complexe K is assembled from $\pi_1(K)$, $\pi_2(K)$, and the k-invariant $\kappa \in H^3(\pi_1(K), \pi_2(K))$.

For G free or $G = \mathbb{Z} \times \mathbb{Z}$, $H^3(G, M) = 0$ for all $\mathbb{Z}G$-modules M. $\pi_2(K)$ is stably free since the cohomological dimension of G is less or equal to two, and hence free by results of Bass (1960) and Suslin-Quillen (1976). It follows that the homotopy type is determined by the Euler characteristic.

Projective modules for the Klein bottle group

Artamonov (1981) studied K-theoretic properties of the solvable groups. Applied to the Klein bottle group G defined by the presentation $F/N = \langle a, b \mid aba^{-1} = b^{-1} \rangle$ one obtains obtains:

Theorem. Let and $\alpha_n = a + 1 + nb^{-1} + nb^{-3}$ and $\beta_n = 1 + nb + nb^3$. Then α_n and β_n generate the left module $\mathbb{Z}G$ and the set $\{\mathcal{K}_{\alpha_n,\beta_n}\}$ contains infinitely many homotopically distinct algebraic 2-complexes.

Generators for $\mathbb{Z}G$

In fact, if $p(x)$ is a polynomial in $\mathbb{Z}[x]$, then $\alpha = a + p(b^{-1})$ and $\beta = p(b)$ generate $\mathbb{Z}G$ as a left module. Note that

$$\begin{aligned}
(a - p(b))\alpha + p(b^{-1})\beta &= \\
(a - p(b))(a + p(b^{-1}) + p(b^{-1})p(b) &= \\
a^2 + ap(b^{-1}) - p(b)a - p(b)p(b^{-1}) + p(b^{-1})p(b) &= a^2
\end{aligned}$$

Algebraic 2-complexes for the Klein bottle group

The algebraic 2-complexes $\{\mathcal{K}_{\alpha_n,\beta_n}\}$ were studied by Misseldine and myself (2010).

- Let $r = aba^{-1}b$. Is

  $$\langle a, b \mid (ara^{-1})(r)(br^{-n}b^{-1})(b^3r^{-n}b^{-3}), (r)(br^nb^{-1})(b^3r^nb^{-3})\rangle$$

  a presentation for G?
- Is any of the $\mathcal{K}_{\alpha_n,\beta_n}$ geometrically realizable?

Thank You

Thanks to the organizers of Groups - St Andrews 2013.

And thank you for listening.