# Involution Statistics in Coxeter Groups

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# A few Statistics

We can associate various numbers to a permutation $\sigma$ of $\mathrm{Sym}(n)$.

- $\mathrm{inv}(\sigma) = |\{(i, j) : 1 \leq i < j \leq n, \sigma(i) > \sigma(j)\}|$, the *inversion number*

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- $\mathrm{des}(\sigma) = |D(\sigma)| = |\{i \in [n] : \sigma(i) > \sigma(i+1)\}|$, the *number of descents*
- $\mathrm{exc}(\sigma) = |\{(i \in [n] : \sigma(i) > i\}|$, the *excedance*
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There are relationships between these statistics, for example (Stanley, Enumerative Combinatorics Corollary 4.5.9):

$$|\{\sigma \in \mathrm{Sym}(n) : \mathrm{inv}(\sigma) = k\}| = |\{\sigma \in \mathrm{Sym}(n) : \mathrm{maj}(\sigma) = k\}|$$

We look at inversion number. (Joint work with Peter Rowley.)

# Inversions

- The number of inversions is a measure of how 'disordered' a permutation is.
- $\mathrm{inv}(Id) = 0$.
- Highest possible is $\binom{n}{2}$, for $\begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}$.
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# Inversion Polynomial

We have (eg Stanley, Enumerative Combinatorics)

$$\sum_{\sigma \in \mathrm{Sym}(n)} t^{\mathrm{inv}(\sigma)} = (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{n-1}).$$

This is symmetric, unimodal, log-concave, nice.

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- Second: $\mathrm{Sym}(n)$ is just one example of a Coxeter group. What is the generalisation of inversion number in that context?

# Coxeter Groups

- A Coxeter system $(W, R)$ is a group $W$ with $W = \langle R \rangle$, and the only relations are $(rs)^{m_{rs}} = 1$ for all $r, s \in R$, such that $m_{rr} = 1$ and $m_{rs} = m_{sr} \geq 2$ when $r \neq s$.

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- Examples: $\mathrm{Sym}(n)$ with $R = \{(12), (23), \ldots, (n-1\ n)\}$
- The dihedral group of order $2m$, where $R$ consists of two reflections whose product has order $m$.
- Every finite Coxeter group is a direct product of irreducible Coxeter groups, and these have been classified.

# The Length Function

Let $W$ be a Coxeter group, and $w \in W$. Then

$$\ell(w) = \min\{k : w = r_1 r_2 \cdots r_k, r_i \in R\}.$$

- For $w \in \mathrm{Sym}(n)$ (which is a Coxeter group of type $A_{n-1}$) it can be shown that
$$\ell(w) = \mathrm{inv}(w).$$

Length is of fundamental importance in Coxeter groups, essentially because of its relationship with the root system $\Phi$ – the length of $w$ is the number of positive roots taken negative by $w$.

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- If $X = W$, then $f_{W,X}(t)$ is known as the Poincaré polynomial $W(t) = \sum_{w \in W} t^{\ell(w)}$. It has been extensively studied. For example, if $W$ is finite irreducible of rank $n$, then there are positive integers $e_1, \ldots, e_n$ (the exponents of $W$) such that

$$W(t) = \prod_{i=1}^{n} (1 + t + \cdots + t^{e_i}).$$

  For $A_n$ the exponents are $1, 2, \ldots, n$.

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- The answers seem nice!

# Groups of Permutations, and the rest

- Groups with a (useful) permutation interpretation of some kind:

$$A_n, B_n, D_n, \tilde{A}_n, \tilde{B}_n, \tilde{C}_n, \tilde{D}_n.$$

- The finite Coxeter groups are well known. Having done types $A_n$, $B_n$ and $D_n$, we then get all the finite Coxeter groups, by reducing to the irreducible case, sorting types $E_6$, $E_7$, $E_8$, $F_4$, $H_3$, $H_4$ by brute force, and doing $I_2(m)$ in our sleep.

# Conjectures

Various conjectures have been made concerning the unimodality and/or log-concavity of the coefficients of involution length polynomials in the classical groups. A sequence $(x_i)_{i=1}^N$ is unimodal if for some $i$ we have $a_1 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_N$ and log-concave if for all $i$ between 2 and $N-1$ we have $x_i^2 \geq x_{i-1}x_{i+1}$. A log-concave sequence of positive integers is always unimodal, but not vice versa.

- (Brenti) The coefficients of $f_{\mathrm{Sym}(n),\mathcal{I}}(t)$ are log-concave (where $\mathcal{I}$ is the set of all involutions of $\mathrm{Sym}(n)$).
- (Dukes) Let $W$ be of type $A, B$ or $D$. Write $\mathcal{I}_e$ (respectively $\mathcal{I}_o$) for the set of even length (respectively odd length) involutions in $W$. Then the sequences of coefficients of both $f_{W,\mathcal{I}_e}(t)$ and $f_{W,\mathcal{I}_o}(t)$ are symmetric and unimodal.

## $A_n$, $B_n$, $D_n$

- $B_n$ is the group of permutations $w$ of $\{1, 2, \ldots, n, -1, -2, \ldots, -n\}$ with the property that $w(-i) = -w(i)$ for all $i$. So for example $(\overset{-}{1}\overset{+}{2}\overset{-}{3})$ means $w(1) = -2$, $w(-1) = 2$, $w(2) = 3$, $w(-2) = -3$ and so on.

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- $A_{n-1} \cong \mathrm{Sym}(n)$ is the subgroup of $B_n$ consisting of permutations with a $+$ sign above every number.
- $D_n$ is the subgroup of $B_n$ whose elements have an even number of sign changes.
- A generating set for $B_n$ is $\{(\overset{-}{1}), (\overset{+}{1}\overset{+}{2}), \ldots, (n \overset{+}{-} 1\ \overset{+}{n})\}$.
- A generating set for $A_{n-1}$ is $\{(\overset{+}{1}\overset{+}{2}), \ldots, (n \overset{+}{-} 1\ \overset{+}{n})\}$.
- A generating set for $D_n$ is $\{(\overset{-}{1}\overset{-}{2}), (\overset{+}{1}\overset{+}{2}), \ldots, (n \overset{+}{-} 1\ \overset{+}{n})\}$.

# Lengths and involutions

- If $w \in D_n$, its length as an element of $D_n$ is likely to be different from its length as an element of $B_n$. For example $(\bar{1}\bar{2})$ has length 1 in $D_n$ but length 3 in $B_n$. Luckily, if $w \in A_{n-1}$ we have $\ell_A(w) = \ell_B(w)$, and if $w \in D_n$ we have $\ell_D(w) = \ell_B(w) - \#(\text{ minus signs in } w)$.
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- With care, we can mostly work in $B_n$.
- Involutions in $W$ have cycles $(\overset{+}{i}\,\overset{+}{j})$, $(\overset{-}{i}\,\overset{-}{j})$, $(\overset{+}{i})$ or $(\overset{-}{i})$.

# Plan of Attack

- Consider an involution $w$ of length $\ell$ in $W_n$ (where $W_n$ is one of $A_{n-1}, B_n$ or $D_n$), with $m$ transpositions and $e$ negative 1-cycles.
- The cycle $\tau$ involving $n$ could be $(\overset{+}{n})$, $(\overset{-}{n})$, $(\overset{+}{k}\overset{+}{n})$ or $(\overset{-}{k}\overset{-}{n})$.
- If $(\overset{+}{n})$ or $(\overset{-}{n})$ then let $z = w\tau$ '$\in$' $W_{n-1}$.
- Otherwise, let $z = (w\tau)^{(n\ n-1\ \cdots k+1\ k)}$. Then $z \in W_{n-2}$.

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For example if $w = (\overset{-}{1}\overset{-}{3})(\overset{-}{2}\overset{-}{5})(\overset{-}{4})$ in $B_5$, then $k = 2$, $w\tau = (\overset{-}{1}\overset{-}{3})(\overset{-}{4})$ and $z = (\overset{-}{1}\overset{-}{2})(\overset{-}{3}) \in W_{n-2}$.
Crucially, $\ell(w) - \ell(z)$ depends *only* on $k$.

# Recursive Formulae

For example, if we write the $P_{n,m,\ell}$ for the set of involutions in $A_{n-1}$ of length $\ell$ having $m$ transpositions, it turns out that

$$P_{n,m,\ell} = P_{n-1,m,\ell} + \sum_{k=1}^{n-1} P_{n-2,m-1,\ell-2(n-k)+1}.$$

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Instead of writing $f_{W,X}$ we write $f_{A_{n-1},m}$ for $X$ the set of involutions in $A_{n-1}$ with $m$ transpositions and $f_{W,m,e}$ for $X$ the set of involutions in $W$ with $m$ transpositions and $e$ negative 1-cycles. If $e$ is odd such involutions are not in $D_n$ but we can still calculate their '$D$-length'.

# Results

**Theorem (Hart, Rowley)**

- $f_{A_{n-1},m}(t) = f_{A_{n-2},m}(t) + \frac{t(t^{2n-2}-1)}{t^2-1} f_{A_{n-3},m-1}(t)$;

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- $f_{B_n,m,e}(t) =$
  $f_{B_{n-1},m,e}(t) + t^{2n-1} f_{B_{n-1},m,e-1}(t) + \frac{t(t^{4n-4}-1)}{t^2-1} f_{B_{n-2},m-1,e}(t)$;

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- $f_{D_n,m,e}(t) = f_{D_{n-1},m,e}(t) + t^{2n-2} f_{D_{n-1},m,e-1}(t) + \frac{t(1+t^{2n-4})(t^{2n-2}-1)}{t^2-1} f_{D_{n-2},m-1,e}(t)$.

$A_n$ result stated by J. Désarménien (1982), cited by Dukes (2007). Closed formula results in all three cases when $n = 2m$, giving symmetric, unimodal sequences. Combining for all $m$, $e$ gives results for set of all involutions.

# Addressing the Conjectures

(For brevity, the sequence of nonzero coefficients of $f_{W,X}$ is called the length profile of $X$ in $W$.)

- These are the only examples we know of for even or odd involution length profiles in finite irreducible Coxeter groups that are not unimodal:
  - Even length involutions in $B_6$.
  - Even length involutions in $E_8$, $F_4$, $H_4$ and $I_2(m)$, for $m$ even.

In particular, Dukes' conjecture is partially false. However, we can refine it.

**Conjecture**

**(i)** *If $X$ is a conjugacy class of involutions in $A_n$ or $B_n$, then the even/odd length profile of $X$ is unimodal.*

**(ii)** *If $X$ is the set of involutions of odd length in a finite Coxeter group, then the (odd) length profile of $X$ is unimodal.*

# What about $\tilde{A}_n$ etc?

Let $G_n$ be the subgroup of permutations $w$ of $\mathbb{Z}$ that satisfy $w(i+n) = w(i) + n$, for all $i \in \mathbb{Z}$, and $\sum_{i=1}^{n} w(i) = \binom{n+1}{2}$. Then $G_n \cong \tilde{A}_{n-1}$.

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- We think(!)

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$$f_{G_n,m}(t) = f_{G_{n-1},m}(t) + \frac{t}{1-t^2} f_{G_{n-2},m-1}(t).$$

- Proof: