Schur Indices in GAP

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The Wedderburn Decomposition

The *Wedderburn decomposition* of a semisimple group algebra FG is its direct sum decomposition

$$FG \simeq \bigoplus_{\chi} M_{r_\chi}(D_\chi)$$

where the components are matrix rings over division algebras D_χ that are finite dimensional over F.

The GAP package `wedderga` (by Broche Cristo, Konovalov, Olteanu, Oliviera, del Río, - later van Gelder, and soon, H.) gives a Wedderburn decomposition of FG, when G is a finite group and F is a field supported by GAP, whose simple components are given in terms of matrix rings over *cyclotomic algebras*:

Example 1: After `gap> LoadPackage("wedderga")` ...

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gap> G:=SmallGroup(48,18);; R:=GroupRing(Rationals,G);;
gap> W:=WedderburnDecompositionInfo(R);
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[2,Rationals],[1,Rationals,3,[2,2,0]],[2,CF(3)],
[1,Rationals,6,[2,5,0]],[1,NF(8,[1,7]),8,[2,7,4]],
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Problem: How do we isolate the division algebra parts of these simple components? Can we get GAP to do it?

The Schur index

Let $A = FGe_\chi = [s, F(\chi), n, \alpha(,\delta)]$ be a simple component of FG corresponding to $\chi \in Irr(G)$. A is a simple algebra, so $A \simeq [r, D]$, for some finite dimensional division algebra D.

The *Schur index* of D (or A or χ over F) is
$m(D) = m(A) = m_F(\chi) = \sqrt{[D : Z(D)]}$.
It measures the dimension of non-commutative part of D. The dimension of A over its center $Z(D) = F(\chi)$ is $r^2 m_F(\chi)^2$.

The Schur Index

Let A be a simple component of the group algebra FG with $d = \sqrt{[A : Z(A)]}$.

Our problem boils down to:
- calculating the Schur index m of A, and
- describing a division algebra D for which $A \simeq [\frac{d}{m}, D]$.

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If F is a finite field, m is always 1. So we assume F is an abelian number field and $[F : \mathbb{Q}] < \infty$.

Local Schur indices

Fact: If K is an algebraic number field, the Schur index of a central simple K-algebra A is

$$m(A) = \underset{\mathcal{P}}{LCM}\{m(A \otimes_K K_{\mathcal{P}})\},$$

where $K_{\mathcal{P}}$ denotes the completion of K at the prime $\mathcal{P}$, and $\mathcal{P}$ runs over the set of all (finite and infinite) primes of K.

Fortunately, if $A = FGe_{\chi}$ is a simple component of FG when F is an abelian number field and G is a finite group, $K = \mathbb{Q}(\chi)$ is an abelian number field, and the local indices $m(A \otimes_K K_{\mathcal{P}})$ agree at primes of K that lie over the same rational prime p.
This local index at p can be > 1 only if $p = \infty$ or p divides $|G|$.

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So to calculate the Schur index $m(A)$ in our situation, we (only) need to calculate the local indices at finitely many rational primes.
Fact: The list of local indices *almost* determines D up to isomorphism.

Command: `SchurIndex(A);` `SchurIndexByCharacter(F,G,n);`

Procedure: Similar to that of MAGMA's `SchurIndex(F,chi)`, which was contributed by Nebe-Unger (2009) and improved for absolute number fields $F = \mathbb{Q}[x]/(f(x))$ by Fieker (2011?).

The important steps (for the local index at p algorithm) are:
(i). explicit Brauer-Witt reductions;
(ii-a). Benard's formula for the local index at p when χ lies in a block with cyclic defect group;
(ii-b). Riese and Schmid's classification of dyadic Schur groups to compute local index at p in other cases; and
(iii). adjusting the local indices when F is larger than the field of character values.

`wedderga`'s new Schur index algorithm

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The code for `wedderga`'s algorithm has been written entirely independently. It computes Schur indices over any abelian number field F and makes use of special case shortcuts.

Suppose $A = $ [s,F,n,[a,b,c]] is a cyclic cyclotomic algebra.

Case 1: If A is a cyclic cyclotomic algebra, its local indices at ∞, 2, and odd p are computed using three shortcut algorithms:

- the local index at ∞ is computed directly from the cyclic cyclotomic algebra presentation without using the Frobenius-Schur indicator;

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- the local index at ∞ is computed directly from the cyclic cyclotomic algebra presentation without using the Frobenius-Schur indicator;

- the local indices at 2 or an odd prime p are computed directly from the cyclic cyclotomic algebra presentation using methods based on Janusz (PJM78). These require wedderga's new *cyclotomic reciprocity functions* for $F(\zeta_n)/F$:

K:=PSplitSubextension(F,n,p),
g:=SplittingDegreeAtP(F,n,p),
f:=ResidueDegreeAtP(F,n,p), and
e:=RamificationIndexAtP(F,n,p).

Cyclic cyclotomic examples

Example 2: Q_8 and $Q_{12} \simeq C_3 \rtimes C_4$.

```
gap> G:=SmallGroup(8,4);; R:=GroupRing(Rationals,G);;
gap> A:=WedderburnDecompositionInfo(R)[5];
[1,Rationals,4,[2,3,2]]
gap> LocalIndicesOfCyclicCyclotomicAlgebra(A);
[[2,2],[infinity,2]]
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gap> LocalIndicesOfCyclicCyclotomicAlgebra(A);
[[2,2],[infinity,2]]
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```
gap> G:=SmallGroup(12,1);;
R:=GroupRing(Rationals,G);;
gap> A:=WedderburnDecompositionInfo(R)[5];
[1,Rationals,6,[2,5,3]]
gap> LocalIndicesOfCyclicCyclotomicAlgebra(A);
[[3,2],[infinity,2]]
```

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gap> LocalIndicesOfCyclicCyclotomicAlgebra(A);
[[2,2],[infinity,2]]
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```
gap> G:=SmallGroup(12,1);;
R:=GroupRing(Rationals,G);;
gap> A:=WedderburnDecompositionInfo(R)[5];
[1,Rationals,6,[2,5,3]]
gap> LocalIndicesOfCyclicCyclotomicAlgebra(A);
[[3,2],[infinity,2]]
gap> CyclotomicAlgebraWithDivAlgPart(A);
[1, rec( Center := Rationals, DivAlg := true,
Local Indices:=[[3,2],[infinity,2]], SchurIndex:=2)]
```

Case 2: A is a simple component `[s,F,n,`α,δ`]` for which the abelian Galois group of $F(\zeta_n)/F$ requires 2 or more generators.

Step 1: Use the presentation of A to produce a defining polycyclic group G_1 (always cyclic-by-abelian) and faithful irreducible character $\chi_1 =$ `Irr(`G_1`)[s]`, for which $FG_1e_{\chi_1}$ is Morita equivalent to A.

If F is real, compute the local index of A at ∞ using the Frobenius-Schur indicator of χ_1: if $\chi_1 =$ `Irr(G)[s]` then this is `Indicator(CharacterTable(G),2)[s]`.

To compute the local index at a finite prime p:
Step 2: Find the maximal p-split subextension K of $F(\zeta_n)/F$.
Reduce from A to $A_1 = KG_1e_{\chi_1}$, this has the same local index at p.

A_1 is often cyclic cyclotomic, if so use the shortcuts as in Case 1.
If A_1 is not cyclic cyclotomic, recalculate its defining group G_2 and character χ_2, and proceed to Step 3.

Schur index algorithm: Cyclic defect group case

Step 3: Calculate the conjugacy classes of possible defect groups of the p-block of G_2 containing $\chi_2 = \texttt{Irr}(G_2)\texttt{[n]}$ using $\texttt{PossibleDefectGroups}(G_2, n, p)$;

If the defect group is cyclic, then by a theorem of Benard (AM76):

$$m_{\mathbb{Q},p}(\chi_2) = [\mathbb{F}_p(\chi_2^o, \phi) : \mathbb{F}_p(\chi_2^o)],$$

for any $\phi \in IBr(G_2)$ lying in the same p-block as χ_2.

If F is larger than the field of character values, a theorem of Yamada shows that

$$m_{F,p}(\chi_2) = m_{\mathbb{Q},p}(\chi_2)/[gcd(m_{\mathbb{Q},p}(\chi_2), e(F/\mathbb{Q}(\chi_2), p)f(F/\mathbb{Q}(\chi_2), p)].$$

If this defect group is not cyclic, we have to do something else.

Schur index algorithm: non-cyclic defect group case

Suppose the defect group of the block containing χ_2 is not cyclic.

We require two assumptions (which can be removed by the "one prime at a time" approach and a norm reduction [H.,CA95]):
(i). That the Galois group for A_1 is 2-generated.
(ii). That (G_2, χ_2) is a terminal Brauer-Witt reduction for p.

If these hold, then the local index at p is 1 unless $p = 2$ and χ_2 is a faithful irreducible character of G_2.

Step 4: Use Riese and Schmid's classification of dyadic Schur groups. The assumptions imply that $m_{\mathbb{Q},2}(\chi_2) = 2$ if and only if G_2 is a dyadic Schur group, and otherwise it will be 1.

Finally, adjust from $m_{\mathbb{Q},2}(\chi_2)$ to $m_{F,2}(\chi_2)$ using the theorem of Yamada.

Back to Example 1:

```
gap> G:=SmallGroup(48,18);; F:=Rationals;;
gap> W:=WedderburnDecompositionInfo(GroupRing(F,G));
[[1,Rationals],[1,Rationals],[1,Rationals],[1,Rationals],
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gap>
WedderburnDecompositionWithDivAlgParts(GroupRing(F,G));
[
[1,Rationals],[1,Rationals],[1,Rationals],[1,Rationals],
[2,Rationals],[2,Rationals],[2,CF(3)],[2,Rationals],
[1,rec(Center:=NF(8,[1,7]), DivAlg:=true,
LocalIndices:=[[infinity,2]], SchurIndex:=2) ],
[2,rec(Center:=Rationals, DivAlg:=true,
LocalIndices:=[[infinity,2],[3,2]], SchurIndex:=2) ];
```

Schur index algorithm: Examples

Example 3:

```
gap> G:=SmallGroup(80,15);; F:=Rationals;;
gap> Size(Irr(G));
17
gap> SimpleComponentOfGroupRingByCharacter(F,G,17);
[1, NF(5,[1,4]), 20, [[2,9,15],[2,11,0]], [[15]] ]
gap> K:=PSplitSubextension(NF(5,[1,4]),20,5);
NF(20,[1,9])
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gap> K:=PSplitSubextension(NF(5,[1,4]),20,5);
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gap> SimpleComponentOfGroupRingByCharacter(K,G,17);
[2, NF(20,[1,9]), 20, [2,9,15]]
gap> LocalIndicesOfCyclicCyclotomicAlgebra(last);
[[5,2]]
```

Schur index algorithm: Examples

Example 4:

```
G:=SL(2,11);; F:=Rationals;;
gap> A:=SimpleComponentOfGroupRingByCharacter(F,G,10);
[1/2,NF(12,[1,11]),132,[[10,73,0],[2,23,66]],[[0]]]
```

Schur index algorithm: Examples

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gap> A:=SimpleComponentOfGroupRingByCharacter(F,G,10);
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gap> CyclotomicAlgebraWithDivAlgPart(A);
[5,rec(Center:=NF(12,[1,11]),DivAlg:=true,
LocalIndices:=[[infinity,2]], SchurIndex:=2) ]
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[5,rec(Center:=NF(12,[1,11]),DivAlg:=true,
LocalIndices:=[[infinity,2]], SchurIndex:=2) ]

gap> SchurIndexByCharacter(Rationals,G,10);
2
```

wedderga's other Schur index features

1. Algorithms for computing local and global Schur indices of generalized quaternion algebras over $\mathbb{Q}$ - based on the traditional Legendre symbol algorithm.

2. Conversions between cyclic cyclotomic algebras and cyclic algebras - up to Morita equivalence. Schur indices of cyclic algebras can be computed by solving relative norm equations, which can be done using PARI.

3. Decomposition of cyclotomic algebras with 2-generated Galois groups into a tensor product of two cyclic algebras - up to Morita equivalence.

4. Conversions between cyclic algebras and generalized quaternion algebras, whenever possible.

The End...Thank you...See you next year!