WEAK CAYLEY TABLES OF GROUPS AND GENERALIZED CENTRALIZER RINGS OF FINITE GROUPS

Stephen Humphries and Emma Rode

Frobenius and the group determinant

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Frobenius also defined the k-characters of $G, k \geq 1$: here 1-characters are just the ordinary characters of G and 2-characters were defined by

$$\chi^{(2)}(g,h) = \chi(g)\chi(h) - \chi(gh).$$

The group determinant determines the group

Theorem (Formanek and Sibley, Mansfield) The group determinant of G determines G.

Theorem (Johnson and Hoenke) The 1-,2- and 3-characters of G determine G.

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This condition implies that G and H have the same character tables.

Weak Cayley Table results

It is known that for a group G the information in each of the following is the same:

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(2) the 1- and 2-characters of G;
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Let $k \geq 1$. Then S_k acts on G^k and G acts on G^k by diagonal conjugation.

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Let $O_1, \dots, O_s$ be the orbits for the action of $\langle S_k, G \rangle$ on G^k.
Then $\{\overline{O_1}, \dots, \overline{O_s}\}$ is a basis for a subring $C^{(k)}(G)$ of $\mathbb{C}G^k$ called the *k-S-ring* of G.

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Example: the 1-S-ring of G is just $Z(\mathbb{C}G)$.

S-rings

Recall: An S-ring over H is a subring of $\mathbb{C}H$ determined by a partition $H = H_1 \cup \cdots \cup H_r$ of H where
(i) $H_1 = \{1\}$;
(ii) for all $i \leq r$ there is $j \leq r$ with $H_i^{-1} = H_j$;
(iii) for all $i, j \leq r$ we have $\overline{H_i}\,\overline{H_j} = \sum_k \lambda_{ijk} \overline{H_k}$ where $\lambda_{ijk} \in \mathbb{Z}^{\geq 0}$.

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We will say that G and H have the *same* $k-S$*-ring* if there is a bijection $\phi : G \to H$ that determines an S-ring isomorphism $C^{(k)}(G) \to C^{(k)}(H)$.

Results on k-S-rings and WCT

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Logical independence of two conditions:

Theorem There are groups which have the same weak Cayley table, but not the same 2-S-rings (e.g. $|G| = p^3$ where p is odd).
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Corollary The 2-S-ring does not determine the group.

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Theorem If there is a bijection $\phi : G \to H$ that is a weak Cayley table isomorphism and determines an isomorphism of 2-S-rings, then $\phi(G^{(i)}) = H^{(i)}$. In particular G and H have the same derived lengths and the same derived series sizes.

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Theorem There are non-isomorphic groups of order 2^9 which have the same weak Cayley table and the same 2-S-rings. They form a Brauer pair.

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Theorem Let G be an FC group and suppose that we know each product xy for all $x, y \in G$ that are not conjugate. Then we can determine the multiplication table of G algorithmically.

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Theorem (Rode thesis 2012) A finite group is determined by $C^{(3)}(G)$.

Results on 3-S-rings

Theorem If $C^{(3)}(G)$ is commutative, then for all ordered pairs $x, y \in G$ we have one of:
(1) $xy = yx$;
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Hypothesis (*): conclusion of above theorem.

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Theorem Hypothesis (*) implies one of
(i) G is abelian;
(ii) G is the generalized dihedral group of an abelian group N of odd order, i.e. $G = N \rtimes \mathcal{C}_2$ where $\mathcal{C}_2$ is the cyclic group of order 2 and its generator conjugates elements of N to their inverses;
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