Commuting probability and commutator relations

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Groups St Andrews 2013

Commuting probability

Let G be a *finite* group. The probability that a randomly chosen pair of elements of G commute is called the **commuting probability** of G.

$$\mathrm{cp}(G) = \frac{|\{(x, y) \in G \times G \mid [x, y] = 1\}|}{|G|^2}$$

- $\mathrm{cp}(G) = \mathrm{k}(G)/|G|$ Erdös, Turán 1968

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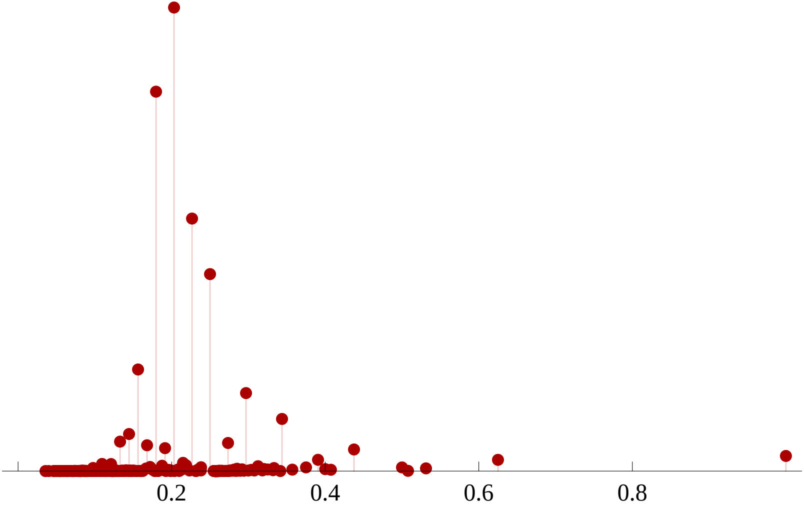
Outlook

Global	Analyse the image of cp.
Local	Study the impact $\mathrm{cp}(G)$ has on the structure of G.

Commuting probability globally

As a function on groups of order ≤ 256



Commuting probability globally

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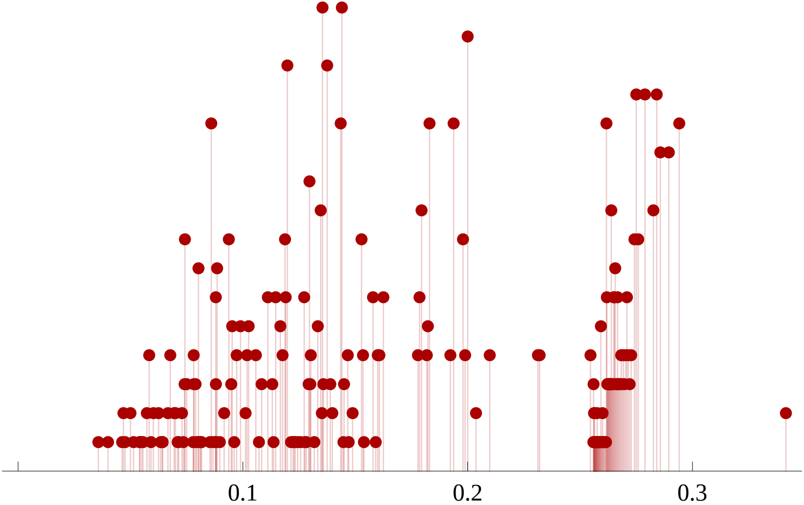

Commuting probability globally

Joseph's conjectures

Conjecture (Joseph 1977)

1. The limit points of $\operatorname{im}\operatorname{cp}$ are rational.
2. If ℓ is a limit point of $\operatorname{im}\operatorname{cp}$, then there is an $\varepsilon > 0$ such that $\operatorname{im}\operatorname{cp} \cap (\ell - \varepsilon, \ell) = \emptyset$.
3. $\operatorname{im}\operatorname{cp} \cup \{0\}$ is a closed subset of $[0, 1]$.

- 1. and 2. hold for limit points $> 2/9$. Hegarty 2012

Commuting probability locally

As a measure of being abelian

- If $\mathrm{cp}(G) > 5/8$, then G is abelian. — Gustafson 1973
- If $\mathrm{cp}(G) > 1/2$, then G is nilpotent. — Lescot 1988
- $\mathrm{cp}(G) < |G : \mathrm{Fit}(G)|^{-1/2}$ — Guralnick, Robinson 2006

Commuting probability locally

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General principle

Bounding $\mathrm{cp}(G)$ away from zero ensures abelian-like properties of G.

Commuting probability locally

Setting up the terrain

The **exterior square** $G \wedge G$ of G is the group generated by the symbols $x \wedge y$ for all $x, y \in G$, subject to *universal commutator relations*:

$$x \wedge x = 1, \quad xy \wedge z = (x^y \wedge z^y)(y \wedge z), \quad x \wedge yz = (x \wedge z)(x^z \wedge y^z).$$

Commuting probability locally

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The **curly exterior square** $G \curlywedge G$ of G is the group generated by the symbols $x \curlywedge y$ for all $x, y \in G$, subject to *universal commutator relations, but without redundancies*, i.e.

$$G \curlywedge G = \frac{G \wedge G}{\langle x \wedge y \mid [x, y] = 1 \rangle}.$$

Commuting probability locally

Bogomolov multiplier

There is a natural commutator homomorphism $\kappa\colon G \curlywedge G \to [G, G]$.

The kernel of κ consists of non-universal commutator relations. This is the **Bogomolov multiplier** of the group G, denoted by $\mathrm{B}_0(G)$.

Commuting probability locally

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The group $\mathrm{B}_0(G)$ is isomorphic to the unramified Brauer group of G, an obstruction to Noether's problem of stable rationality of fixed fields.

- $\mathrm{Br}_{nr}(\mathbb{C}(G)/\mathbb{C})$ embeds into $\mathrm{H}^2(G, \mathbb{Q}/\mathbb{Z})$. Bogomolov 1987
- The image of the embedding is $\mathrm{B}_0(G)^*$. Moravec 2012

Commuting probability locally

Bogomolov multiplier: examples

$B_0 = 0$

- Abelian-by-cyclic groups — Bogomolov 1988
- Finite simple groups — Kunyavskiĭ 2010
- Frobenius groups with abelian kernel — Moravec 2012
- p-groups of order $\leq p^4$ — Bogomolov 1988
- Most groups of order p^5 — Moravec 2012
- Unitriangular p-groups

$B_0 \neq 0$

- Smallest possible order is 64. — Chu, Hu, Kang, Kunyavskiĭ 2010
- $\langle a, b, c, d \mid [a, b] = [c, d], \text{exp } 4, \text{cl } 2 \rangle$

Commuting probability locally

The general principle universally

Theorem

If $\mathrm{cp}(G) > 1/4$*, then* $\mathrm{B}_0(G) = 0$.

Commuting probability locally

The general principle universally

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Outline of proof

Assume that G is a group of the smallest possible order satisfying $\mathrm{cp}(G) > 1/4$ and $\mathrm{B}_0(G) \neq 0$. By standard arguments, G is a stem p-group.

Proper subgroups and quotients of G have a larger commuting probability than G, so: $\mathrm{B}_0(G) \neq 0$*, but all proper subgroups and quotients of* G *have a trivial Bogomolov multiplier*. Groups with the latter property are called B_0-**minimal**.

B_0-minimal groups

A B_0-minimal group enjoys the following properties.

- Is a capable p-group with an abelian Frattini subgroup.
- Is of Frattini rank ≤ 4.
- For stem groups, the exponent is bounded by a function of class alone.

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- Is a capable p-group with an abelian Frattini subgroup.
- Is of Frattini rank ≤ 4.
- For stem groups, the exponent is bounded by a function of class alone.

- Given the nilpotency class, there are only finitely many isoclinism families containing a B_0-minimal group of this class.
- Classification of B_0-minimal groups of class 2, hence of class 2 groups of orders p^7 with non-trivial Bogomolov multipliers.
- Construction of a sequence of 2-groups with non-trivial Bogomolov multipliers and arbitrarily large nilpotency class.

Commuting probability locally

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Considering the structure of B_0-minimal groups of coclass 3, use the class equation to obtain bounds on the sizes of conjugacy classes of a *suitably* chosen generating set of G. This restricts the nilpotency class of G. Finish with the help of NQ.