Old and new developments in group matrices

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Outline

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Group matrices

Let G be a finite group of order n with a listing of elements $\{g_1 = e, g_2, ..., g_n\}$ and let $\{x_{g_1}, x_{g_2}, ..., x_{g_n}\}$ be a set of independent commuting variables indexed by the elements of G.

Definition

The (full) *group matrix* X_G is the matrix whose rows and columns are indexed by the elements of G and whose $(g, h)^{\text{th}}$ entry is $x_{gh^{-1}}$.

The group matrix is a patterned matrix: it is determined by its first row (or column)

Example

The group matrix of C_3 is (abbreviating x_{g_i} by i) the circulant

$$C(1,2,3) = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}.$$

Further example

Example

The group matrix of S_3 is the matrix

$$\begin{bmatrix} 1 & 3 & 2 & 4 & 5 & 6 \\ 2 & 1 & 3 & 6 & 4 & 5 \\ 3 & 2 & 1 & 5 & 6 & 4 \\ 4 & 6 & 5 & 1 & 2 & 3 \\ 5 & 4 & 6 & 3 & 1 & 2 \\ 6 & 5 & 4 & 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} C(1,2,3) & C(4,6,5) \\ C(4,5,6) & C(1,3,2) \end{bmatrix}$$

group matrices obtained from the cosets of an arbitrary subgroup

If $|G| = kr$ and H is any cyclic subgroup of order k then the elements of G can be listed such that X_G is a block matrix of the form

$$\begin{bmatrix} B_{11} & B_{12} & ... & B_{1r} \\ B_{21} & B_{22} & ... & B_{2r} \\ ... & .. & ... & .. \\ B_{r1} & B_{r2} & ... & B_{rr} \end{bmatrix},$$

where each B_{ij} is a circulant of size $k \times k$. A corresponding result holds for any subgroup H. (Dickson 1907) If in the above H is arbitrary, X_G is as above, but the blocks are now all of the form $X_H(g_{i_1}, g_{i_2} ... g_{i_k})$. Here elements in the vector $(g_{i_1}, g_{i_2} ... g_{i_k})$ are elements in G, and not necessarily arising from any specific coset of H.

Example for arbitrary subgroup

Example

Let $G = \mathcal{S}_4$ and $H =< (1,2,3,4),(1,4)(2,3) >$ be a copy of D_8. With the ordering of G on the cosets of H, $X_G = \{B_{ij}\}_{i,j=1}^3$ where each B_{ij} is of the form $X_H(\underline{u}_{i,j})$ with

$$\underline{u}_{1,1} = (1,2,3,4,5,6,7,8) \qquad \underline{u}_{1,2} = (11,12,22,15,13,24,10,9)$$
$$\underline{u}_{1,3} = (9,23,16,18,21,11,20,14) \qquad \underline{u}_{2,1} = (9,10,11,12,13,14,15,16)$$
$$\underline{u}_{2,2} = (1,20,6,23,21,8,18,3) \qquad \underline{u}_{2,3} = (17,7,24,2,5,19,4,22)$$
$$\underline{u}_{3,1} = (17,18,19,20,21,22,23,24) \qquad \underline{u}_{3,2} = (9,4,14,7,5,16,2,11)$$
$$\underline{u}_{3,3} = (1,11,2,16,5,7,14,4)$$

Dickson's results on the mod p case

The group determinant mod p of a p-group.

Lemma

Let H be any p-group of order $r = p^s$. Let P be the upper triangular matrix of the form

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ & 1 & 2 & 3 & & r-1 \\ & & 1 & 3 & & (r-1)(r-2)/2 \\ & & & 1 & & \dots \\ & & & & \dots & r-1 \\ & & & & & 1 \end{bmatrix}.$$

Then a suitable ordering of H exists such that, modulo p, PX_HP^{-1} is a lower triangular matrix with identical diagonal entries of the form $\alpha = \sum_{i=1}^{r} x_{h_i}$.

The group determinant Θ_H modulo p is thus α^r.

Example

$G = C_5$. Then $P =$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 & 4 \\ & & 1 & 3 & 6 \\ & & & 1 & 4 \\ & & & & 1 \end{bmatrix}$$

and modulo 5

$$PX_GP^{-1} = \begin{bmatrix} \alpha & 0 & 0 & 0 & 0 \\ \beta & \alpha & 0 & 0 & 0 \\ \gamma & \beta & \alpha & 0 & 0 \\ \delta & \gamma & \beta & \alpha & 0 \\ \mu & \delta & \gamma & \beta & \alpha \end{bmatrix}$$

where $\alpha = \sum_{i=1}^{5} x_{g_i}$, $\beta = 4x_2 + 3x_3 + 2x_4 + x_5$, $\gamma = x_2 + 3x_3 + x_4$, $\delta = 4x_2 + x_3$ and $\mu = x_2$.

Question: does this have any relevance to the FFT?

Lemma

Let G be a group of order n divisible by p and H be a Sylow-p subgroup of index k and order r. Then, an ordering of G exists such that, modulo p, X_G is similar to a matrix which has a block diagonal part of the form

$$diag(B, B, ..., B) \text{ (r occurences of B)}$$

with the upper triangular part above the diagonal 0. Moreover B encodes the permutation representation of G on the cosets of H.

This is proved by acting on the X_G obtained by ordering G by the left cosets of H and acting by $diag(P, P, ..., P)$ and rearranging. Thus it follows that, modulo p, $\Theta_G = \det(B)^r$.
Question: is there an explanation of all this using the standard techniques of modular representation theory?

Superalgebras

Superalgebras arose in physics. A superalgebra is a $\mathbb{Z}_2$-graded algebra, i.e. it is an algebra over a commutative ring or field with a decomposition into "even" and "odd" pieces, with a multiplication operator which respects the grading. More formally
Let K be a commutative ring. A *superalgebra* over K is a K-module A with a direct sum decomposition

$$A = A_0 \oplus A_1$$

with a bilinear multiplication $A \times A \to A$ such that

$$A_i A_j \subset A_{i+j}$$

where the subscripts are read mod 2.

Superalgebras continued

Usually, K is taken to be $\mathbb{R}$ or $\mathbb{C}$. The elements of A_i, $i = 1, 2$ are said to be *homogeneous.* The *parity* of a homogeneous element x, denoted by $|x|$, is 0 or 1 depending on whether it is in A_0 or A_1. Elements of parity 0 are said to be *even* and those of parity 1 are said to be *odd*. If x and y are both homogeneous, then so is the product and $|xy| = |x| + |y|$.

A superalgebra is *associative* if its multiplication is associative. It is *unital* if it has a multiplicative identity, which is necessarily even. It is usual to assume that superalgebras are both associative and unital.

A superalgebra A is *commutative* if for all homogeneous $x, y \in A$,

$$yx = (-1)^{|x||y|}xy.$$

The standard example is an exterior algebra over K. Another example is the algebra A of symmetric and alternating polynomials, with A_0 the symmetric polynomials and A_1 being the alternating polynomials.

Supermatrices

Definition

Let R be a superalgebra, which is unital and associative. Let p, q, r, s be nonnegative integers. A *supermatrix* of dimension $(r|s) \times (p|q)$ is an $(r+s) \times (p+q)$ matrix X with entries in R which is partitioned into a 2×2 block structure

$$X = \begin{bmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{bmatrix},$$

so that X_{00} has dimensions $r \times p$ and X_{11} has dimensions $s \times q$. An ordinary (ungraded) matrix may be interpreted as a supermatrix with $q = s = 0$.

Definition

A square supermatrix X has $(r|s) = (p|q)$.

This implies that X, X_{00} and X_{11} are all square in the usual sense.

Even and odd supermatrices

An *even supermatrix* X has diagonal blocks X_{00} and X_{11} consisting of even elements of R, and X_{01} and X_{10} consisting of odd elements of R, i.e. it is of the form

$$\begin{bmatrix} even & odd \\ odd & even \end{bmatrix}.$$

An *odd supermatrix* X has diagonal blocks which are odd and the remaining blocks even, i.e. it is of the form

$$\begin{bmatrix} odd & even \\ even & odd \end{bmatrix}.$$

If the scalars R are purely even then there are no nonzero odd elements, so the even supermatrices are the block diagonal ones

$$X = \begin{bmatrix} X_{00} & 0 \\ 0 & X_{11} \end{bmatrix},$$

Even and odd supermatrices continued

and an odd supermatrix is of the form

$$X = \begin{bmatrix} 0 & X_{01} \\ X_{10} & 0 \end{bmatrix}.$$

A supermatrix is *homogenous* if it is either even or odd. The *parity*, $|X|$, of a non-zero homogeneous supermatrix X is 0 or 1 according to whether it is even or odd. Every supermatrix can be written uniquely as the sum of an even matrix and an odd one.

Operations

Let X, Y be supermatrices. $X+Y$ is defined entrywise, so that

$$X + Y = \begin{bmatrix} X_{00} + Y_{00} & X_{01} + Y_{01} \\ X_{10} + Y_{10} & X_{11} + Y_{11} \end{bmatrix}.$$

The sum of even matrices is even, and the sum of odd matrices is odd.

XY is defined by ordinary block matrix multiplication, i.e.

$$XY = \begin{bmatrix} X_{00}Y_{00} + X_{01}Y_{10} & X_{00}Y_{01} + X_{01}Y_{11} \\ X_{10}Y_{00} + X_{11}Y_{10} & X_{10}Y_{01} + X_{11}Y_{11} \end{bmatrix}.$$

If X and Y are both even or both odd, then XY is even, and if they differ in parity XY is odd.

The scalar multiplication differs from the ungraded case. It is necessary to define left and right scalar multiplication. If $\widehat{\alpha} = (-1)^{|\alpha|}\alpha$ *left scalar multiplication* by $\alpha \in R$ is defined by

$$\alpha.X = \begin{bmatrix} \alpha X_{00} & \alpha X_{01} \\ \widehat{\alpha} X_{10} & \widehat{\alpha} X_{11} \end{bmatrix},$$

$$X.\alpha = \begin{bmatrix} X_{00}\alpha & X_{01}\widehat{\alpha} \\ X_{10}\alpha & X_{11}\widehat{\alpha} \end{bmatrix}.$$

If α is even then $\widehat{\alpha} = \alpha$ and both operations are the same as the ungraded versions. If α and X are homogeneous, then both $\alpha.X$ and $X.\alpha$ are homogeneous with parity $|\alpha| + |X|$. If R is supercommutative, then $\alpha.X = (-1)^{|\alpha||X|}X.\alpha$.

supertranspose

The supertranspose of the homogeneous supermatrix X is the $(p|q) \times (r|s)$ supermatrix

$$X^{st} = \begin{bmatrix} X_{00}^t & (-1)^{|X|} X_{10} \\ -(-1)^{|X|} X_{01} & X_{11}^t \end{bmatrix}$$

where M^t denotes the usual transpose of a matrix. This can be extended to arbitrary supermatrices by linearity. The supertranspose is not an involution: if X is an arbitrary supermatrix, then

$$(X^{st})^{st} = \begin{bmatrix} X_{00} & -X_{01} \\ -X_{10} & X_{11} \end{bmatrix}.$$

If R is supercommutative then for arbitrary supermatrices X, Y

$$(XY)^{st} = (-1)^{|X||Y|} Y^{st} X^{st}.$$

Parity Transpose

There is a new operation, the *parity transpose*. This is denoted by X^π. If X is a supermatrix, then

$$X^\pi = \begin{bmatrix} X_{11} & X_{10} \\ X_{01} & X_{00} \end{bmatrix},$$

and the following are satisfied

$$(X+Y)^\pi = X^\pi + Y^\pi,$$

$$(XY)^\pi = X^\pi Y^\pi,$$

$$(\alpha.X)^\pi = \widehat{\alpha}.X^\pi,$$

$$(X.\alpha)^\pi = X^\pi.\widehat{\alpha}$$

and in addition

$$\pi^2 = 1$$

$$\pi \circ st \circ \pi = (st)^3.$$

Supertrace and Berezinian

The supertrace of a square supermatrix is defined on homogeneous supermatrices by the formula

$$str(X) = tr(X_{00}) - (-1)^{|X|} tr(X_{11}).$$

If R is supercommutative then

$$str(XY) = (-1)^{|X||Y|} str(YX).$$

for homogeneous supermatrices X, Y.
The *Berezinian* or *superdeterminant* $\mathrm{Ber}(X)$ of a square supermatrix X is only well-defined on even invertible supermatrices over a commutative superalgebra R. In this case

$$\mathrm{Ber}(X) = \det(X_{00} - X_{01} X_{11}^{-1} X_{10}) \det(X_{11})^{-1},$$

where det denotes the ordinary determinant of square matrices with entries in the commutative algebra R_0. The Berezinian satisfies similar properties to the ordinary determinant. In particular, it is multiplicative and invariant under the supertranspose.

Moreover

$$\mathrm{Ber}(e^X) = e^{str(X)}.$$

In particular, if R is purely even and X is even, then

$$Ber(X) = \det(X_{00}) \det(X_{11})^{-1}.$$

The ring of virtual representations

Let $\mathrm{Irr}(G) = \{\chi_i\}_{i=1}^r$and take supermatrices of the form

$$\overline{X}_G = \begin{bmatrix} X_{1G} & 0 \\ 0 & X_{2G} \end{bmatrix},$$

where X_{1G} and X_{2G} are group matrices. The supertrace of $\overline{X}_G$ is $tr(X_{1G}) - tr(X_{2G})$. Then

$$\mathrm{Ber}(\overline{X}_G) = \det(X_{1G})\det(X_{2G})^{-1}$$

where det is the ordinary determinant.
Consider a virtual representation of G with generalized character

$$\psi = \sum_{i=1}^{r} s_i\chi_i - \sum_{i=1}^{r} t_i\chi_i$$

For any character $\sum_{i=1}^{r} s_i \chi_i$ of G there is naturally associated the group matrix $\sum_{g \in G} \sum_{i=1}^{r} s_i \rho_i(g)$. Denote this by $X_G^{\sum s_i \rho_i}$. Then associate to ψ the super group matrix

$$\begin{bmatrix} X_G^{\sum s_i \rho_i} & 0 \\ 0 & X_G^{\sum t_i \rho_i} \end{bmatrix}.$$

The ring of virtual group representations may be obtained by factoring out by the equivalence relation $\equiv$ on arbitrary group supermatrices defined by

$$\begin{bmatrix} X_{1G} & 0 \\ 0 & X_{2G} \end{bmatrix} \equiv \begin{bmatrix} \widehat{X}_{1G} & 0 \\ 0 & \widehat{X}_{2G} \end{bmatrix}$$

if and only if X_{1G} is similar to $X_G^{\sum s_i \rho_i}$, X_{1G} is similar to $X_G^{\sum t_i \rho_i}$, $\widehat{X}_{1G}$ is similar to $X_G^{\sum \widehat{s}_i \rho_i}$, $\widehat{X}_{2G}$ is similar to $X_G^{\sum \widehat{t}_i \rho_i}$ such that $s_i - t_i = \widehat{s}_i - \widehat{t}_i$ for $i = 1...r$

Additional ideas

A random walk on a group G associated to a probability p on G.
Equivalent to a Markov chain with transition matrix $X_G(p)$ (obtained by replacing x_g by $p(g)$ for all $g \in G$)

If p is constant on conjugacy classes, then $X_G(p)$ can be diagonalised
(equivalent to the specialised version of X_G having linear factors)

The question can be formulated in terms of S-rings (Wielandt)

Call an S-ring S on a group a **fission** if the classes of S are obtained by splitting the conjugacy classes. If S is commutative it gives rise to a diagonalised X_G.

Result (Humphries):
The maximum number of classes in an S-ring S giving rise to a diagonalised X_G is $\tau(G) = \sum_{\chi \in Irr(G)} \deg(\chi)$ (the dimension of a Gelfand model).
Strange fact: the Jucy's Murphy elements in the group ring of the symmetric group produce a commutative subring of the group ring of dimension $\tau(G)$, but this is not an S-ring