Variations on a theme of I.D. Macdonald

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I.D. Macdonald, *On cyclic commutator subgroups*, J. London Math. Soc. 38 (1963), pp. 419-422.

Theorem 1. *If G' is cyclic and either G is nilpotent or G' is infinite, then G' is generated by a suitable commutator. For any given positive integer n, however, there is a group G in which G' is cyclic and is generated by no set of less than n commutators.*

The general question:

Given a word $f(x_1, \ldots, x_k)$ in the variables $x_1, x_2, \ldots, x_k$, let

$$F(G) = \{f(g_1, \ldots, g_k); g_1, \ldots, g_k \in G\},$$

the set of values of f in a group G, and $f(G) = \langle F(G) \rangle$, the word subgroup of f in G.

Question 1. Under what conditions is $f(G) = F(G)$?

Question 2. Let $f(G) = \langle c \rangle$. Under what conditions do we have $c \in F(G)$?

Question 3. Let $f(G) = \langle c \rangle$ with $c \in F(G)$. Is $f(G) = F(G)$?

Notation. $K(G) = \{[a,b] \big| a,b \in G\}$, the set of commutators in G. ($K(G) = F(G)$ for $f(x,y) = [x,y]$).

D.M. Rodney, *On cyclic derived subgroups*, J. London Math. Soc. (2) 8 (1974), pp. 642-643.

Theorem 2. *There exist groups G with $G' = \langle [a,b] \rangle$ but $G' \neq K(G)$. If the automorphism group induced on G' is cyclic, then $G' = K(G)$.*

Words of the lower central series:

$f(x_1, \ldots, x_n) = [x_1, x_2, \ldots, x_n]$

$f(G) = G_n, \; F(G) = K_n(G).$

L.-C. Kappe, *Groups with a cyclic term in the lower central series*, Arch. Math. 30 (1978) pp. 561-569.

Theorem 3. *If G_n is cyclic and either G is nilpotent or G_n is infinite cyclic, then G_n is generated by a suitable commutator of weight n. For any given integer m, however, there is a group G in which G_n is cyclic and generated by no set of less than m commutators of weight n.*

Powers:

$f(x) = x^n$, n an integer > 0;
$F(G) = G^{(n)}$, $f(G) = G^n$.

L.C. Kappe and G. Mendoza, *Groups of minimal order which are not n-Power closed*, AMS Contemporary Math. 511 (2010), pp. 93-107.

Classification of minimal counterexamples to the conjecture $G^{(n)} = G^n$ for given $n > 1$, $n = 2^\alpha \cdot k, \alpha = 0, 1, 2, 3, k$ odd.

Theorem 4. *Let n be an odd integer and p the smallest prime dividing n. Then $G^{(n)} = G^n$ for all G with $|G| \leq 2p$, except for*

$$G \cong D_p = \langle a, b | a^p = b^2 = 1, a^b = a^{-1} \rangle \ ,$$

the dihedral group of order $2p$, for which $G^n \neq G^{(n)}$.

Example: There exist groups G and integers $n > 1$ with G^n cyclic but G^n cannot be generatd by the n-th power of an element in G:

Let p and q be distinct primes, $q|p-1$, α, β, γ integers with $\alpha, \beta \geq 2$ and $\gamma \not\equiv 1 \mod p^{\alpha}$ but $\gamma^{q} \equiv 1 \mod p^{\alpha}$. Let
$G = \langle a, b; a^{p^{\alpha}} = b^{q^{\beta}} = 1,\ a^{b} = a^{\gamma} \rangle$.
Then $G^{p^{\sigma} q^{\tau}}$ is cyclic, $1 \leq \sigma < \alpha, 1 \leq \tau < \beta$, but not generated by an element in $G^{(p^{\sigma} q^{\tau})}$.

Remark 1. $G^{p^{\sigma} q^{\tau}}$ is generated by two elements in $G^{(p^{\sigma} q^{\tau})}$. Are there groups G and integers n such that G^{n} cyclic but every generator of G^{n} is the product of more than two elements in $G^{(n)}$?

Remark 2. Obviously $G^n = \langle g^n \rangle$ always implies $G^n = G^{(n)}$.

Remark 3. We are looking for sufficient conditions on n or G such that G^n cyclic implies $G^n = \langle g^n \rangle$.

Theorem 5. *Let G be a group of square-free order. Then G^n cyclic implies $G^n = \langle g^n \rangle$.*

Theorem 6. *Let G be locally nilpotent with G^n cyclic. Then there exists $g \in G$ such that $G^n = \langle g^n \rangle$.*

The case G^n infinite cyclic:

Theorem 7. *Let G be a group and n an integer with G^n infinite cyclic. Then there exists $g \in G$ with $G^n = \langle g^n \rangle$ provided one of the following conditions holds:*

(i) G *is locally nilpotent;*

(ii) $n = p^m$, p *a prime;*

(iii) G/G^n *is locally finite;*

(iv) $G^n \subseteq Z(G)$.

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