On finite groups with small prime spectrum

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Definitions

Let G be a finite group.

Denote by $\pi(G)$ the set of prime divisors of the order of G.
If $|\pi(G)| = n$ then G is called n-primary.

$\Gamma(G)$ is a prime graph of the group G; in this graph, the vertex set is $\pi(G)$ and two different vertices p and q are adjacent if and only if in G there is an element of order pq.

$s = s(G)$ denotes the number of connected components of the graph $\Gamma(G)$.

$\{\pi_i(\mathrm{G}) \mid 1 \leq \mathrm{i} \leq \mathrm{s}\}$ denotes the set of all connected components of the graph $\Gamma(G)$.
If $2 \in \pi(G)$ then $2 \in \pi_1(G)$.

Recognition groups by prime graph

A group G is called recognizable by prime graph if, for any finite group H, the graph equality $\Gamma(H) = \Gamma(G)$ implies a group isomorphism $H \cong G$.

The first example of groups recognizable by prime graph were provided by M. Hagie in 2003, namely some sporadic simple groups.

The problem of recognition of a group by prime graph is a particular case of more general problem: study finite groups by the properties of their prime graphs.

Groups with disconnected prime graph

We mainly consider groups with disconnected prime graph

The class of finite groups with disconnected prime graph
1) generalizes widely the class of finite Frobenius groups;
2) coincides with the class of finite groups having an isolated subgroup;
3) includes important "small" groups.

Groups with disconnected prime graph

Suppose that $\Gamma(G)$ is disconnected.
Let $\overline{G} = G/F(G)$ be an almost simple group.

There exists $g \in G \backslash F(G)$ such that $|g| \in \pi(\overline{G})$ and g acts freely on p-chief factors of the group G, $p \in \pi(F(G))$.

Problem

Let G be a finite group, Q be a normal nontrivial subgroup from G, $\overline{G} = G/Q$ be a known group and an element of prime order from $G \setminus Q$ acts on Q fixed-points-freely. The following questions arise.
1) What are the chief factors of the group G in Q?
2) What is the structure of the group Q?
3) If Q is elementary abelian group, is the action of $\overline{G}$ on Q completely irreducible?
4) Is the extension of G over Q splittable?

1. G. Higman (1968): $\overline{G} \cong L_2(2^n), n \geq 2$, element of order 3 acts fixed points freely on Q.
2. R.P. Martineau (1972): $\overline{G} \cong Sz(2^n)$, n is odd, element of order 5 acts fixed points freely on Q.
3. W.B. Steward (1973): $\overline{G} \cong L_2(q)$, q is odd, element of order 3 acts fixed points freely on Q.
4. A. Prince (1977), G. Zurek (1982), D.F. Holt and W. Plesken (1986): $\overline{G} \cong A_5$, $Q = O_2(G)$, element of order 5 acts fixed points freely on Q.
5. A. Prince (1982): $\overline{G} \cong A_6$, $Q = O_2(G)$, element of order 5 acts fixed points freely on Q.

Groups with small prime spectrum

We investigate the first item of the problem for groups with small prime spectrum.

In 2010 — 2013 Kondratiev and Khramtsov described the chief factors of 3-primary and most part of 4-primary groups with disconnected prime graph.

4-primary groups

Theorem (Kondratiev, Khramtsov, 2011) *Let G be a finite 4-primary group with disconnected prime graph, and let $\overline{G} = G/F(G)$. Then, one of the following statements holds:*
(1) G is a Frobenius group;
(2) G is a 2-Frobenius group;
(3) $\overline{G}$ is an almost simple triprimary group;
(4) $\overline{G} \cong L_2(2^m)$, where $m \geq 5$, $2^m - 1$, and $(2^m + 1)/3$ are primes;
(5) $\overline{G} \cong L_2(3^m)$ or $PGL_2(3^m)$, where m and $(3^m - 1)/2$ are odd primes and $(3^m + 1)/4$ is either a prime or 11^2 (for $m = 5$);
(6) $\overline{G} \cong L_2(r)$ or $PGL_2(r)$, where r is a prime, $17 \neq r \geq 11$, $r^2 - 1 = 2^a 3^b s^c$, $s > 3$ is a prime, $a, b \in \mathbb{N}$, and c is either 1 or 2 for $r \in \{97, 577\}$;

4-primary groups

(7) $\overline{G} \cong A_7$, S_7, A_8, S_8, A_9, $L_2(16)$, $L_2(16)\colon 2$, $\mathrm{Aut}(L_2(16))$, $L_2(25)$, $L_2(25)\colon 2$, $L_2(27)\colon 3$, $L_2(49)$, $L_2(49)\colon 2_1$, $L_2(49)\colon 2_3$, $L_2(81)$, $L_2(81)\colon 2$, $L_2(81)\colon 4$, $L_3(4)$, $L_3(4)\colon 2_1$, $L_3(4)\colon 2_3$, $L_3(5)$, $\mathrm{Aut}(L_3(5))$, $L_3(7)$, $L_3(7)\colon 2$, $L_3(8)$, $L_3(8)\colon 2$, $L_3(8)\colon 3$, $\mathrm{Aut}(L_3(8))$, $L_3(17)$, $\mathrm{Aut}(L_3(17))$, $L_4(3)$, $L_4(3)\colon 2_2$, $L_4(3)\colon 2_3$, $U_3(4)$, $U_3(4)\colon 2$, $\mathrm{Aut}(U_3(4))$, $U_3(5)$, $U_3(5)\colon 2$, $U_3(7)$, $\mathrm{Aut}(U_3(7))$, $U_3(8)$, $U_3(8)\colon 2$, $U_3(8)\colon 3_1$, $U_3(8)\colon 3_3$, $U_3(8)\colon 6$, $U_3(9)$, $U_3(9)\colon 2$, $\mathrm{Aut}(U_3(9))$, $U_4(3)$, $U_4(3)\colon 2_2$, $U_4(3)\colon 2_3$, $U_5(2)$, $\mathrm{Aut}(U_5(2))$, $S_4(4)$, $S_4(4)\colon 2$, $\mathrm{Aut}(S_4(4))$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_4(9)\colon 2_1$, $S_4(9)\colon 2_3$, $S_6(2)$, $G_2(3)$, $\mathrm{Aut}(G_2(3))$, $O_8^+(2)$, ${}^3D_4(2)$, $\mathrm{Aut}({}^3D_4(2))$, $Sz(8)$, $Sz(32)$, $\mathrm{Aut}(Sz(32))$, ${}^2F_4(2)'$, ${}^2F_4(2)$, M_{11}, M_{12}, $\mathrm{Aut}(M_{12})$, or J_2.

Recognizable groups with small prime spectrum

As a corollary, the following results were obtained.

Corollary 1 *A finite* 3*-primary almost simple group with disconnected prime graph is recognizable by prime graph if and only if it is isomorphic to* $L_2(17)$.

Corollary 2 *A finite* 4*-primary simple group is recognizable by prime graph if and only if it is isomorphic to one of the following groups:* A_8, $L_3(4)$, *and* $L_2(q)$, *where* $|\pi(q^2-1)| = 3$, $q > 17$, *and either* $q = 3^m$ *and* m *is an odd prime or* q *is a prime and* $q \not\equiv 1 \pmod{12}$ *or* $q \in \{97, 577\}$.

Theorem for A_7

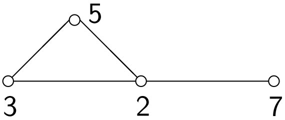
We investigated the problem in the case when $Q = O_2(G)$, $\overline{G} \cong A_7$ and an element of order 5 from G acts on Q fixed points freely. The following theorem was proved.

Theorem (Kondratiev, Khramtsov, 2012) *Let G be a finite group with a nontrivial normal 2-subgroup Q and $G/Q \cong A_7$. Suppose that an element of order 5 from G acts on Q fixed points freely. Then the extension G over Q is split, Q is an elementary abelian group and Q is the direct product of minimal normal subgroups each of which as $GF(2)G/Q$-module is isomorphic to one of the two 4-dimensional irreducible $GF(2)A_7$-modules that are conjugated by outer automorphism of the group A_7.*

Finite groups with the same graph as the group $Aut(J_2)$

Khosravi in 2009 obtained a description of a group having the same prime graph as the group $Aut(S)$ for any sporadic simple group S except for the group J_2. He posed the problem: *describe all groups G such that* $\Gamma(G) = \Gamma(Aut(J_2))$. It should be noted that $\Gamma(Aut(J_2))$ is connected. Kondratiev solved the Khosravi's problem in 2012.

$\Gamma(Aut(J_2))$:



Finite groups with the same graph as the group A_{10}

The group A_{10} is exceptional in some senses. It is the only group with connected prime graph among all finite simple groups from "Atlas of finite groups"and also among all 4-primary simple groups. Recently Kondratiev described all finite group with the same prime graph as the group A_{10}.

$\Gamma(A_{10})$: