Recent advances on prime graphs of integral group rings

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Joint work with V. Bovdi, E. Jespers, W. Kimmerle, S. Linton, S. Siciliano et al.

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Conjectures

Let $V = V(\mathbb{Z}G)$ be the normalised unit group of the integral group ring $\mathbb{Z}G$ of a finite group G.

The following conjectures about elements of V state that:

- **(ZC-1)**: they are rationally conjugate to elements of G (Zassenhaus, 1974).
- **(IP-C)**: they have same orders as elements of G.
- **(PQ)**: if G has elements of prime orders p and q but no elements of order pq, then V has no elements of order pq (Kimmerle, Oberwolfach Reports, 2007).

Motivation

- **(ZC-1)** holds for nilpotent groups (Weiss, 1991)
- **(PQ)** holds for solvable groups (Kimmerle, 2006)
- Clearly, **(ZC-1)** $\Rightarrow$ **(IP-C)** $\Rightarrow$ **(PQ)**
- For a particular group G, all three conjectures involve looking at various possible orders of torsion units of $V(\mathbb{Z}G)$
- Motivated by this, jointly with V. Bovdi we started a project of determination of properties of torsion units of $V(\mathbb{Z}G)$ for sporadic simple groups

Toolkit

Partial augmentations

sums of coefficients of elements of $\mathbb{Z}G$ over conjugacy classes

Criterion for ZC-1

formulated in terms of partial augmentations

HeLP (*He*rtweck-*L*uthar-*P*assi) method

uses character tables to produce constraints on partial augmentations

Reducing to a finite problem

Finitely many possible orders of torsion units

The order of $u \in V(\mathbb{Z}G)$ divides $exp(G)$
(Cohn–Livingstone, 1965)

Finitely many possible partial augmentations

$$\nu_i(u)^2 \leq |C_i| \quad \text{and} \quad \sum_{i=1}^{n} \frac{\nu_i(u)^2}{|C_i|} \leq 1$$

(Hales–Luthar–Passi, 1990)

- $|\nu_{5a}| \leq 39$ for M_{11}
- $|\nu_{30a}| \leq 58\,023\,609\,591\,071\,951\,707\,573\,011$ for M

Reducing the number of search variables

- 10 conjugacy classes of elements in M_{11}
- 194 in M

Berman–Higman Theorem (1955)

$$\mathrm{tr}(u) = \nu_1 = 0$$

2005–2006: Hertweck generalised results by Marciniak–Ritter–Sehgal–Weiss (1987) and Luthar–Passi (1989)

$$\nu_g(u) \neq 0 \Rightarrow o(g) \quad \text{divides} \quad o(u)$$

- Now only 2 search variables for order 10 in M_{11}
- ... but 28 search variables for order 30 in the Monster

Main source of constraints

Theorem (Luthar–Passi, 1989; modular case - Hertweck, 2005)

for all l the number

$$\mu_l(u, \chi, p) = \frac{1}{k} \sum_{d|k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}}\Big(\chi(u^d) z^{-dl}\Big)$$

is a non-negative integer which is not greater than $\deg(\chi)$*, where:*

- p *is either* 0 *or a prime divisor of* $|G|$
- $u \in V(\mathbb{Z}G)$ *is a normalized torsion unit of order* k
- *if* $p \neq 0$*, then* k *and* p *must be coprime*
- z *is a complex primitive* k*-th root of unity*
- χ *is a classical character or a* p*-Brauer character of* G

Example: order 77 for Co_1

$$\mu_{11}(u,\chi_2,0) = \tfrac{1}{77}(-100\nu_{7a} - 30\nu_{7b} - 10\nu_{11a} + 10\nu_{11a}^{(7)} - 10\nu_{7a}^{(11)} - 3\nu_{7b}^{(11)} + 276) \geq 0;$$

$$\mu_{0}(u,\chi_3,0) = \tfrac{1}{77}(300\nu_{7a} + 300\nu_{7b} + 120\nu_{11a} + 20\nu_{11a}^{(7)} + 30\nu_{7a}^{(11)} + 30\nu_{7b}^{(11)} + 299) \geq 0;$$

$$\mu_{0}(u,\chi_4,0) = \tfrac{1}{77}(840\nu_{7a} + 84\nu_{7a}^{(11)} + 1771) \geq 0;$$

$$\mu_{1}(u,\chi_4,0) = \tfrac{1}{77}(14\nu_{7a} - 14\nu_{7a}^{(11)} + 1771) \geq 0;$$

$$\mu_{7}(u,\chi_4,0) = \tfrac{1}{77}(-84\nu_{7a} + 84\nu_{7a}^{(11)} + 1771) \geq 0;$$

$$\mu_{11}(u,\chi_4,0) = \tfrac{1}{77}(-140\nu_{7a} - 14\nu_{7a}^{(11)} + 1771) \geq 0;$$

$$\mu_{0}(u,\chi_7,0) = \tfrac{1}{77}(840\nu_{7a} - 120\nu_{11a} - 20\nu_{11a}^{(7)} + 84\nu_{7a}^{(11)} + 27300) \geq 0;$$

$$\mu_{0}(u,\chi_{15},13) = \tfrac{1}{77}(-420\nu_{7a} + 60\nu_{11a} + 10\nu_{11a}^{(7)} - 42\nu_{7a}^{(11)} + 474145) \geq 0;$$

$$\nu_{7a} + \nu_{7b} + \nu_{11a} = 1; \qquad \nu_{7a}^{(11)} + \nu_{7b}^{(11)} = 1 \qquad \nu_{11a}^{(7)} = 1;$$

Results

(PQ) holds for 13 sporadic simple groups:

- M_{11}, M_{12}, M_{22}, M_{23}, M_{24}
- J_1, J_2, J_3
- HS, McL, He, Ru, Suz

Furthermore:

- For $G = ON$, the prime graph of $V(\mathbb{Z}G)$ is not connected
- For $G = Co_3$, Co_2 and Co_1, prime graphs of G and $V(\mathbb{Z}G)$ have the same number of components

Recent overview:

- V. Bovdi, A. Konovalov, S. Linton, *Torsion units in integral group rings of Conway simple groups*, Int. J. Alg. and Comput. 21 (2011), no.4, 615–634

Hot off the press

- (PQ) may be reduced to the examination of nonabelian composition factors and their automorphism groups.
- Theorem (W. Kimmerle, AK, 2012): (PQ) holds for a group of order divisible by three 3 primes, except possibly the case when M_{10} or $PGL(2,9)$ are involved in G.
- A. Bächle and L. Margolis (2013) completed the remaining two cases of M_{10} and $PGL(2,9)$.
- Theorem (W. Kimmerle, AK, 2013): If G is one of M_{12}, M_{22}, J_2, J_3, HS, McL, He, Suz, then (PQ) holds for $Aut(G)$
- Theorem (W. Kimmerle, AK, 2013): (PQ) holds for a finite group provided that each its composition factor S is isomorphic to one of the 13 sporadic simple groups for which (PQ) holds, or $|S|$ divisible by at most three primes.

Some challenges

for $Co3$, this will give a positive answer to (PQ) :

$$|u| = 35 \Rightarrow (\nu_{5a}, \nu_{5b}, \nu_{7a}) \notin \{(3, 12, -14), (4, 11, -14)\}$$

for M_{11}, this will solve (IP-C) :

$$|u| = 12 \Rightarrow (\nu_{2a}, \nu_{4a}, \nu_{6a}) \notin \{(-1, 1, 1), (1, 1, -1)\}$$

Complete all Brauer character tables for $Co1$

and hope to solve (PQ) by eliminating orders 55 and 65

Complete all Brauer character tables for J_4

and cut about 2^{54} admissible triples $(\nu_{31a}, \nu_{31b}, \nu_{31c})$

Find a counterexample to the 1st Zassenhaus conjecture

constructing a unit with prescribed partial augmentations