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Solvability criteria for finite loops and groups

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What is a loop?

Definition

Let Q be a groupoid with a neutral element e. If the equations $ax = b$ and $ya = b$ have unique solutions x and y in Q for every $a, b \in Q$, then we say that Q is a *loop*.

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- If a loop is associative, it is in fact a group.

A small example

1	2	3	4	5
2	1	4	5	3
3	5	1	2	4
4	3	5	1	2
5	4	2	3	1

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- Loops of smaller order are groups.
- $(3 \cdot 4) \cdot 5 \neq 3 \cdot (4 \cdot 5)$
- The analogue of Lagrange's theorem for loops is not true!

Solvability

Definition

A loop Q is said to be *solvable*, if it has a series $1 = Q_0 \subseteq \cdots \subseteq Q_n = Q$, where Q_{i-1} is normal in Q_i and Q_i/Q_{i-1} is an abelian group for each i.

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- Normal subloops are Lagrange-like.
- Our example is simple and non-associative, and thus not solvable!
- Loops of odd order are not always solvable.

Groups associated with loops

Definition

For each $a \in Q$ we have two permutations on Q, defined by $L_a(x) = ax$ and $R_a(x) = xa$ for every $x \in Q$. The *multiplication group* of Q is $M(Q) = \langle L_a, R_a : a \in Q \rangle$. The stabilizer of the neutral element e is $I(Q)$, the *inner mapping group* of Q.

Connected transversals

Definition

Let G be a group, $H \leq G$ and let A and B be two left transversals to H in G. We say that the transversals A and B are *H-connected* if $[A, B] \leq H$.

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- $A = \{L_a : a \in Q\}$ and $B = \{R_a : a \in Q\}$ are $I(Q)$-connected transversals in $M(Q)$, $M(Q) = \langle A, B \rangle$ and $I(Q)_{M(Q)} = 1$.

Two theorems

Theorem [Kepka, Niemenmaa 1990]

A group G is isomorphic to the multiplication group of a loop if and only if there exist a subgroup H and H-connected transversals A and B such that $H_G = 1$ and $G = \langle A, B \rangle$.

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Theorem [Vesanen 1996]

Let Q be a finite loop. If $M(Q)$ is a solvable group, then Q is a solvable loop.

Research question

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Which properties of the inner mapping group $I(Q)$ of a finite loop Q guarantee the solvability of $M(Q)$, and hence that of the loop Q?

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Theorem

Let G be a finite group, $H \leq G$ and let A and B be H-connected transversals in G. If H is of a given structure (next slide), then G is solvable.

If H is

- cyclic (1990),

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If H is

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- $S \times L$, where S is either dihedral or nonabelian of order pq, L is abelian and $(|S|, |L|) = 1$ (2012),

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- $D \times S$, where D is a dihedral 2-group and S is nonabelian of order pq (2013),

then G is solvable.

Theorem and its proof

Theorem (2013)

Let G be a finite group, $H \leq G$ and $H = D \times S$, where D is a Dedekind group, S is a nonabelian group of order pq ($p \neq q$ odd primes) and $\gcd(|D|, |S|) = 1$. If there exist H-connected transversals A and B in G, then G is solvable.

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This claim is also true when G is infinite.

Thank you for your attention!