On Groups with Few Isomorphism Classes of Derived Subgroups

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Introduction

Let G be a group. By a derived subgroup in G is meant the commutator subgroup H' of a subgroup H of G.

We denote by

$$\mathfrak{C}(G)$$

the set of all derived subgroups in the group G:

$$\mathfrak{C}(G) = \{H' \mid H \leq G\}.$$

For example G', $1 \in \mathfrak{C}(G)$.

Question

How important is the subset $\mathfrak{C}(G)$ within the lattice $\mathfrak{S}(G)$ of all subgroups of G?
What are the consequences for the structure of G if conditions are imposed on the set $\mathfrak{C}(G)$?

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A sample result is the following:

Theorem (F. de Giovanni, D.J.S. Robinson; M. Herzog, P. L., M. Maj , 2005)

Let G be a locally graded group.
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Now, if n is a positive integer, let

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denote the class of groups in which the number of the isomorphism classes of derived subgroups is at most n.
Obviously

$$\mathfrak{D}_1 \subseteq \mathfrak{D}_2 \subseteq \ldots \subseteq \mathfrak{D}_n \subseteq \ldots$$

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Examples

• A group G is a $\mathfrak{D}_1$-group if $H' \simeq \{1\}$, for any subgroup H of G.

· Therefore $\mathfrak{D}_1$ is the class of all abelian groups.

• A group G is a $\mathfrak{D}_2$-group if either G is abelian, or $H' \simeq G'$, for any non-abelian subgroup H of G.

· If G' is infinite cyclic or cyclic of order a prime p, then $G \in \mathfrak{D}_2$.

· Free groups of countable rank are in $\mathfrak{D}_2$.

· Groups with all proper subgroups abelian (in particular Tarski groups) are in $\mathfrak{D}_2$.

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· $\mathbb{Q} * \mathbb{Z}$ (a locally free group) is in $\mathfrak{D}_2$.

· A_4, $\mathbb{Z} wr \mathbb{Z}$, $\mathbb{Z}_p wr \mathbb{Z}$ (p a prime) are soluble $\mathfrak{D}_2$-groups.

• If A is a direct product of n copies of a group C where C is infinite cyclic or of order an odd prime p and x inverts the elements of A, then $G = \langle x \rangle \ltimes A$ is in $\mathfrak{D}_n$ and not in $\mathfrak{D}_{n-1}$.

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$\mathfrak{D}_n$-groups and $\mathfrak{D}$-groups

$\mathfrak{D}_1$ is exactly the class of all abelian groups.

A group G is a $\mathfrak{D}_2$-group if either G is abelian, or $H' \simeq G'$, for any non-abelian subgroup H of G.

Proposition

A locally finite $\mathfrak{D}_2$-group G is soluble.

Proof. In fact, assume G not soluble. Then there exist two non-commuting elements $x, y \in G$. Write $H = \langle x, y \rangle$. H is finite since G is locally finite and H is not abelian. Hence H has a minimal non-abelian subgroup K and $G' \simeq K'$. By a classical result of G.A. Miller and H.C. Moreno, K is soluble. Thus G' is soluble and G is soluble, a contradiction. //

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Notice that:

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A soluble $\mathfrak{D}_n$-group has derived length at most n.

Now we can ask:

Problem

What is the maximum integer m such that every locally finite group in $\mathfrak{D}_m$ is soluble?

The answer is $m = 4$.

In fact, we have the following:

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In fact, we have the following:

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A locally finite $\mathfrak{D}_4 - group$ is soluble, but A_5 is a $\mathfrak{D}_5 - group$.

Sketch of the Proof. It is enough to prove the result for a **finite** group. Let G be a finite insoluble $\mathfrak{D}_4$-group of least order.

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Then there are at least five non-isomorphic derived subgroups in G, a contradiction.

On the other hand, A_5 is a $\mathfrak{D}_5$-group since its non-trivial derived subgroups are $A_5, \mathbb{Z}_5, \mathbb{Z}_3, V_4$. //

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Let G be a periodic locally graded group.
G is in $\mathfrak{D}$ if and only if G' is finite.

G is called a locally graded group if every finitely generated non-trivial subgroup of G has a non-trivial finite image.

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A group G is a $\mathfrak{D}_2$-group if either G is abelian, or $H' \simeq G'$, for any non-abelian subgroup H of G.

Examples

· Abelian groups are in $\mathfrak{D}_2$.
· If G' is infinite cyclic or cyclic of prime order p, then $G \in \mathfrak{D}_2$.
· Free groups of countable rank are in $\mathfrak{D}_2$.
· Groups with all proper subgroups abelian (in particular Tarski groups) are in $\mathfrak{D}_2$.

The class $\mathfrak{D}_2$ is *subgroup closed*.

If G is a $\mathfrak{D}_2$-group, N is normal in G and *G' satisfies min, the minimal condition on subgroups*, then *G/N is in $\mathfrak{D}_2$*.

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Nilpotent $\mathfrak{D}_2$-groups admit a very simple description.

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Let G be a non-abelian group. Then G is nilpotent and belongs to $\mathfrak{D}_2$ if and only if G' is cyclic of prime or infinite order and $G' \leqslant Z(G)$.

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Locally finite $\mathfrak{D}_2$-groups

Let G a locally finite $\mathfrak{D}_2$-group. By previous results G is metabelian. Moreover G' is finite.

Example

Let p be a prime and $m > 1$ an integer prime to p. Let $n = |p|_m$ be the order of p modulo m, i.e. the smallest $n > 0$ such that $p^n \equiv 1 \ (mod \ m)$. Let F be a finite field of order p^n. Then F^* has a subgroup $X = \langle x \rangle$ of order m.
Then X acts on $A = F^+$ via the field multiplication and we can define

$$G(p, m) = X \ltimes A,$$

which is a metabelian group of order mp^n.
When $G(p, m) \in \mathfrak{D}_2$?

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We say that (p, m) is an allowable pair if $|p|_m = |p|_d$ for any divisor $d > 1$ of m.

Hence (p, m) is an allowable pair if and only if $G(p, m)$ is a $\mathfrak{D}_2$-group.

Theorem

Let G be a non-nilpotent group with G' finite.
Then G is a $\mathfrak{D}_2$-group if and only if the following hold:
(i) $G = X \ltimes A$ where $A = G'$ is an elementary abelian p-group and $X/C_X(A)$ is cyclic of order m;
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(i) $G = X \ltimes A$ where $A = G'$ is an elementary abelian p-group and $X/C_X(A)$ is cyclic of order m;
(ii) $C_X(A) = Z(G)$, $G/Z(G) \simeq G(p, m)$ and (p, m) is allowable.

Locally finite $\mathfrak{D}_2$-groups

Let p be a prime and $m > 1$ an integer prime to p.
We say that (p, m) is an allowable pair if $|p|_m = |p|_d$ for any divisor $d > 1$ of m.

Hence (p, m) is an allowable pair if and only if $G(p, m)$ is a $\mathfrak{D}_2$-group.

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· If m is a prime, then (p, m) is allowable.

· (p, m) is allowable if and only if $|p|_m = |p|_q$ for all primes q dividing m.

· Let $m = q_1^{e_1} \cdots q_k^{e_k}$ be the primary decomposition of m. Then (p, m) is allowable if and only if each $(p, q_i^{e_i})$ is allowable and $|p|_{q_1} = \cdots = |p|_{q_k}$.
This reduces the problem of finding allowable pairs (p, m) to the case $m = q^e$, with q a prime.

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Lemma

If $q \neq p$ is a prime, then (p, q^e) is allowable if and only if

$$p^{q-1} \equiv 1 \ (mod \ q^e).$$

The previous condition always holds if $e = 1$, by Fermat's little theorem, but rarely if $e > 1$.

Problem

Given a prime p, does there exist a prime q such that

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This is a hard number theoretic problem.

A prime q such that

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A computer search shows that for all $p < 100$, with the possible exception of $p = 47$, there is at least one base-p Wieferich prime.

Only two base-2 Wieferich primes are known,

$$1093 \textit{ and } 3511.$$

There are no others $< 6 \cdot 10^9$.
It is unknown whether infinitely many exist.
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base-p Wieferich primes

Wieferich discovered a connection with Fermat's Last Theorem.
In 1909 he proved that if there is a non-trivial solution of

$$x^q + y^q = z^q$$

with q a prime not dividing xyz, then q is a base 2-Wieferich prime.
One year later Mirimanoff proved that q is also a base-3 Wieferich prime.

There are many interesting open questions concerning base-p Wieferich primes.
An unsolved question is whether is possible for a number to be a base-2 and base-3 Wieferich prime simultaneously.

Soluble $\mathfrak{D}_2$-groups

If G is a soluble $\mathfrak{D}_2$-group, then G' is abelian.

More can be said on the structure of G'.

Proposition

If G is a soluble $\mathfrak{D}_2$-group, then G' is either an elementary abelian p-group for some prime p, or it is free abelian, or it is torsion-free of finite rank.

In fact all three possibilities can occur. Indeed the commutator subgroup of the wreath product $\mathbb{Z}_p$ *wr* $\mathbb{Z}$ is an elementary abelian p-group, the commutator subgroup of the wreath product $\mathbb{Z}$ *wr* $\mathbb{Z}$ is free abelian and the commutator subgroup of the infinite dihedral group is infinite cyclic.

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More can be said about the G-module structure of G'.

When G' is torsion-free of finite rank, a soluble $\mathfrak{D}_2$-group G is constructible up to finite index from an algebraic number field.

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Other classes of infinite $\mathfrak{D}_2$-groups

Theorem

Let G be a perfect group in $\mathfrak{D}_2$. Then G has no proper subgroups of finite index.

Corollary

Let G be a $\mathfrak{D}_2$-group and assume that G' has a proper subgroup of finite index. Then G is a hypoabelian group.

Question

Is a residually finite $\mathfrak{D}_2$-group always residually soluble?

Remark

Let G be a locally graded $\mathfrak{D}_2$-group. If G is periodic, then G is soluble.

Question

What is the structure of a locally graded $\mathfrak{D}_2$-group?

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$\mathfrak{D}_3$-groups

A group G is a $\mathfrak{D}_3$-group if there exist at most three types of non-isomorphic derived subgroups.

Examples

· If $|G'| = p^2$ (p a prime), then G is a $\mathfrak{D}_3$-group.

· If $G = \langle a, x, y \rangle$ is a finite p-group of class 2 and G' is elementary abelian of order p^3, then G is a $\mathfrak{D}_3$-group.
In fact, for every proper non-abelian subgroup X of G either $X' = G'$ or $|X'| = p$.
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· If $G = \langle a, x \rangle$ is a finite p-group of class 3, $p > 2$, G' is elementary abelian of order p^3 and $\{g^p \mid g \in G\} \subseteq Z(G)$, then G is a $\mathfrak{D}_3$-group.

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Then $G \in \mathfrak{D}_3$ if and only if one of the following holds:

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The proof uses some old results on **Camina** p-groups due to **MacDonald** and some recent results due to **M. Lewis**.

Locally finite $\mathfrak{D}_3$-groups

Theorem

Let G be a nilpotent locally finite group.

Then $G \in \mathfrak{D}_3$ if and only if one of the following holds:

(i) $|G'|$ divides p^2 (p a prime);

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Let G be a non-nilpotent locally finite $\mathfrak{D}_3$-group.

Proposition

G' is a finite p-group of nilpotence class at most 2 for some prime p.

Then

$$G/G' = P/G' \times Q/G',$$

where $P = O_p(G)$ and Q/G' is a p'-group.

Proposition

$G = X \ltimes P$ where $P = O_p(G)$ and X is an abelian p'-group for some prime p. Moreover $X/C_X(P)$ is finite.

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The subgroup $O_p(G)$ is abelian or it is an extraspecial p-group, with exponent p if $p > 2$.

First we deal with the groups for which $O_p(G)$ is abelian: once again finite fields play an essential part here.

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Example

Let p be a prime and $m > 1$ an integer prime to p. Let $n = |p|_m$ be the order of p modulo m, i.e. the smallest $n > 0$ such that $p^n \equiv 1$ $(mod\ m)$.
Let F be a finite field of order p^n. Then F^* has a subgroup $X = \langle x \rangle$ of order m.
Then X acts on $A = F^+$ via the field multiplication and we can define

$$G(p, m) = X \ltimes A,$$

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We say that the pair (p, m) is $2-$ *allowable* if, for any $d > 1$ dividing m, it is true that

$$|p|_d = |p|_m \quad or \quad \frac{1}{2}|p|_m.$$

Lemma

The group $G(p, m)$ is a $\mathfrak{D}_3$-group if and only if (p, m) is a 2-allowable pair.

Let A be an X-module. We say that A is *strongly $X -$ simple* if it is a simple Y-module for every non-trivial subgroup Y of X.

For example if X, A are abelian, A is strongly X-simple if and only if $X \ltimes A$ is a $\mathfrak{D}_2$-groups.

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Let G be a locally finite non-nilpotent $\mathfrak{D}_3$ – group with $P = O_p(G)$ abelian. Then

$$G = C \times (X \ltimes P_1)$$

with C abelian, X an abelian p'-group and $P_1 = [P, X]$. In addition one of the following holds:

(i) P_1 is elementary abelian and is a X-simple module, also $G/C_X(P) \simeq G(p, m)$ where (p, m) is 2-allowable;

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