Near supplements and complements in solvable groups of finite rank

Karl Lorensen (speaker)
Pennsylvania State University, Altoona College

Peter Kropholler (coauthor)
University of Southampton

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The cohomology of virtually torsion-free solvable groups of finite rank

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If π is a set of primes, then a solvable minimax group whose spectrum is contained in π is said to be *π-minimax*.

The group-theoretic problem

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Question 1. Under what conditions must G have a π-minimax subgroup X such that $[G : KX] < \infty$?

Question 2. When is it possible to find such a subgroup X with $K \cap X = 1$?

An example where no π-minimax near supplement is present

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However, there is no $\{q\}$-minimax near supplement to K in G.

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It turns out that the answer to this question is yes!

Cohomology and near splittings

Proposition (D. J. S. Robinson)

Assume that G is a group and A a $\mathbb{Z}G$-module that has finite rank as an abelian group. Let ξ be an element of $H^2(G, A)$ and $1 \to A \to E \to G \to 1$ a group extension corresponding to ξ. Then ξ has finite order if and only if E contains a subgroup X such that $X \cap A$ is finite and $[E : AX]$ is finite.

Our cohomology (and homology!) theorem

Theorem A (Kropholler, Lorensen)

Let π be a set of primes and G a virtually torsion-free π-minimax group. Assume that A is a $\mathbb{Z}G$-module whose underlying abelian group is torsion-free and has finite rank. Suppose further that A does not have any nontrivial $\mathbb{Z}G$-module sections that are torsion-free and π-minimax as abelian groups. Then there exists a positive integer m such that

$$m \cdot H^n(G, A) = 0 \quad \text{and} \quad m \cdot H_n(G, A) = 0$$

for all $n \geq 0$.

A property of Ext and Tor

Proposition A (Kropholler, Lorensen, and an anonymous referee)

Let G be an abelian group. Assume that A and B are $\mathbb{Z}G$-modules whose additive groups are torsion-free abelian groups of finite rank. Suppose further that A and B fail to have a pair of nontrivial, mutually isomorphic $\mathbb{Z}G$-module sections that are torsion-free as abelian groups. Then there is a positive integer m such that

$$m \cdot \mathrm{Ext}^n_{\mathbb{Z}G}(A, B) = 0 \quad \text{and} \quad m \cdot \mathrm{Tor}_n^{\mathbb{Z}G}(A, B) = 0$$

for all $n \geq 0$.

Our near supplement theorem

Theorem B (Kropholler, Lorensen)

Let G be a solvable group with finite rank. Assume that K is a normal subgroup of G such that G/K is π-minimax for some set π of primes. If G/K is virtually torsion-free, then there is a π-minimax subgroup X of G such that $[G : KX]$ is finite.

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Corollary

Let G be a solvable group with finite rank. Assume that K is a normal subgroup of G such that G/K is π-minimax for some set π of primes. If G/K is finitely generated, then there is a π-minimax subgroup X of G such that $[G : KX]$ is finite.

The class $\mathfrak{X}_\pi$

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The class $\mathfrak{X}_\pi$ is closed under taking subgroups of finite index, forming homomorphic images, and constructing extensions.

Our near complement theorem

Theorem C (Kropholler, Lorensen)

Let G be a solvable group with finite rank. Assume that K is a normal subgroup of G such that K belongs to the class $\mathfrak{X}_\pi$ and K is Noetherian as a G-operator group. Suppose further that G/K is π-minimax and virtually torsion-free. Then K has a near complement in G.