On groups with given spectrum

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Groups St Andrews, 2013

Intro

- Denote by $\omega(G)$ the spectrum of a group G, i. e. the set of its element orders.
- If the spectrum of G is finite, let $\mu(G)$ be the set of maximal with respect to division elements of $\omega(G)$.

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Burnside problem

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Locally finite group G is a group such that every finite subset of G is contained in a finite subgroup.

- Speaking of a group of period n we mean any group with the identity $x^n = 1$.

Known results. Bounded period

n	lf	G	*Authors*
2	$+$	Abelian	*Burnside*
3	$+$	nilpotent of class 3	*Burnside/ Levi, van der Waerden, 1932*
4	$+$	derived length grows	*Sanov, 1940/Razmyslov, 1978*
5	??		
6	$+$	derived length $\leqslant 4$	*M. Hall, 1958*
$n = 12$, add. cond.	$+$		*Mazurov, Mamontov, 2013*
$n \geqslant 101$, odd n	$-$	ex. of not lf group	*Adian, 2013*
$n \geqslant 8000$	$-$	ex. of not lf group	*Lysönok, 1996*

Known results. Prescribed spectrum

$\mu(G)$	lf	G	*Authors*
$\{2,3\}$	$+$	eA:cyclic	*B. Neumann, 1937*
$\{3,4\}$	$+$	derived length $\leqslant 3$	*Sanov, 1940/Lytkina, 2007*
$\{2,5\}$	$+$	see the structure	*M. Newman, 1979*
$\{2, 2^m+1, 2^m-1\}$, $m \geqslant 2$	$+$	$G \simeq L_2(2^m)$	*Mazurov, Zhurtov, 1999*
$\omega(G)=\{2\}\cup\omega'$, ω' is a set of odds	$\pm$	see the structure	*Mazurov, 2000*
$\{3,5\}$	$+$	see the structure	*Gupta, Maz., 1999/Jabara, 2004*
$\{4,5\}$	$+$	see the structure	*Gupta, Maz., 1999/Jabara, 2004*
$\{3,4,5\}$	$+$	either A_6, or ...	*Mazurov, 2000*
$\{5,6\}$	$+$	soluble	*Mazurov, Mamontov, 2009*
$\{3,4,7\}$	$+$	$G \simeq L_2(7)$	*Lytkina, Kuznetsov, 2007*
$\{3,8\}$	$+$	derived length $\leqslant 4$	*Mazurov, 2010*

New results

Theorem 1 [Jabara and Lytkina, 2013]

Let G be a periodic group with $\mu(G) = \{4, 9\}$. Then G is locally finite and one of the following statements holds:

1. Sylow 2-subgroup T of G is nilpotent of class 2 and normal in G, and a Sylow 3-subgroup is cyclic and acts freely on T.
2. $O_2(G)$ is a non-trivial elementary Abelian group, a Sylow 3-subgroup R of G is cyclic and acts freely on $O_2(G)$.
3. Sylow 3-subgroup R of G is Abelian and normal in G. Sylow 2-subgroup T of G acts freely on R and is either a cyclic group, or a quaternion group. Besides, $G = RT$.

Jabara, E. and Lytkina, D. V. (2013).
On groups of period 36.
Siberian Math. J., 1:29–32.

New results

Theorem 2 [Jabara, Lytkina, Mamontov]

Let G be a periodic group with $\mu(G) = \{3, 5, 8\}$. Then $G \simeq M_{10}$, where $M_{10} = A_6.2$ is a point stabilizer in a sporadic simple Mathieu group M_{11}.

New results

Theorem 3 [Jabara, Lytkina, Mazurov]

Let G be a periodic group with $\mu(G) = \{8, 9\}$. Then G is locally finite.

New results

Theorem 4 [Jabara, Lytkina, Mazurov]

Let G be a locally finite group with $\mu(G) = \{2^m, 3^n\}$ and $m, n \geqslant 1$. Then one of the following statements holds:

- $G = V \rtimes T$, where V is an Abelian 3-group of exponent 3^n and T is either a cyclic 2-group of order 2^m or a generalized quaternion group of order 2^{m+1} acting freely on V;
- $G = V \rtimes T$, where V is a nilpotent 2-group of class at most 2 and exponent 2^m and T is a cyclic 3-group of order 3^n acting freely on V;
- $G = V \rtimes T\langle t\rangle$, where V is a nilpotent 2-group of class at most 2 and exponent 2^{m-1}, T is a 3-group acting freely on V and $T\langle t\rangle$ is isomorphic to a dihedral group of order $2 \cdot 3^n$.

$\mu(G) = \{2, 3\}$

[Neumann, 1937]

Let G be a group with $\mu(G) = \{2, 3\}$. Then G is an extension of elementary Abelian p-group A by a cyclic q-group, acting freely on A. Here $\{p, q\} = \{2, 3\}$.

Neumann, B. H. (1937).
Groups whose elements have bounded orders.
J. London Math. Soc., 12:195–198.

$\mu(G) = \{3, 4\}$

[Lytkina, 2007]

If $\mu(G) = \{3, 4\}$, then G is locally finite [Sanov, 1940] and one of the following holds:

- $G = VQ$, where V is a non-trivial elementary Abelian 3-group, Q is a 2-group, acting freely on V and isomorphic to either C_4, or Q_8;
- $G = T\langle a \rangle$, where T is a normal 2-subgroup of nilpotency class 2, and $|a| = 3$;
- $G = TS$, where T is an elementary Abelian normal 2-group, and $S \simeq S_3$.

In particular, G is soluble of derived length at most 3.

Lytkina, D. V. (2007).
Structure of a group with elements of order at most 4.
Siberian Math. J., 48(2):283–287.

$\mu(G) = \{2, 5\}$

[Newman, 1979]

Suppose that $\mu(G) = \{2, 5\}$. Then G is either an extension of an elementary Abelian 2-group A by a group P of period 5, acting freely on A, or an extension of elementary Abelian 5-group by a group of order 2.

In the first case $|P| = 5$ [Jabara, 2004], thus anyway G is locally finite.

Newman, M. F. (1979).
Groups of exponent dividing seventy.
Math. Scientist, 4:149–157.

$\{2, 2^m + 1, 2^m - 1\}$, $m \geqslant 2$

[Zhurtov and Mazurov, 2009]

Suppose $\mu(G) = \{2, 2^m + 1, 2^m - 1\}$, where $m \geq 2$. Then $G \simeq L_2(2^m)$.

Zhurtov, A. K. and Mazurov, V. D. (2009).
Local finiteness of some groups with given element orders (in russian).
Vladikavkaz Math. Zh., 11(4):11–15.

Groups of period 12

[Lytkina, Mamontov, Mazurov, 2012]

Let G be a group of period 12, such that the product of every two involutions of G is distinct from 6. Then G is locally finite.

[Mamontov, Mazurov, 2013]

Let G be a group of period 12, such that the product of every two involutions of G is distinct from 4. Then G is locally finite.

[Mamontov, 2013]

Groups of period 12 without elements of order 12 are locally finite.

$\omega(G) = \{2\} \cup \omega'$, ω' is a set of odds

[Mazurov, 2000]

Suppose that $\omega(G) = \{2\} \cup \omega'$, where ω' is a set of odds. Then one of the following conditions holds:

- G is an extension of an Abelian group A by a group $\langle t \rangle$ of order 2, and $a^t = a^{-1}$ for every $a \in A$.
- G is an extension of an elementary Abelian 2-group A by an involution-free group, acting freely on A via conjugation in G.
- $G \simeq L_2(Q)$ for some locally finite field Q of characteristic 2.

Groups of the first two forms are locally finite. There exist groups of the latter form which are not locally finite.

Mazurov, V. D. (2000).
Infinite groups with abelian centralizers of involutions.
Algebra and Logic, 39(1):42–49.

$\mu(G) = \{3, 5\}$

[Gupta and Mazurov, 1999]

If $\mu(G) = \{3, 5\}$, then either $G = FT$, where F is a nilpotent of class 2 normal 5-group, and $|T| = 3$, or G is an extension of a 3-group, which is nilpotent of class 3, by a group of order 5. In particular, G is locally finite.

Gupta, N. D. and Mazurov, V. D. (1999).
On groups with small orders of elements.
Bull. Austral. Math. Soc., 60:197–205.

$\mu(G) = \{4, 5\}$

[Gupta and Mazurov, 1999]

If $\mu(G) = \{4, 5\}$, then one of the following conditions holds.

- $G = TD$, where T is a normal elementary Abelian 2-group, and D is a non-abelian group of order 10.
- $G = FT$, where F is an elementary Abelian normal 5-subgroup, and T is isomorphic to a subgroup of Q_8.
- $G = TF$, where T is a nilpoten of class 6 normal 2-group, and F is a group of order 5.

In particular, G is locally finite.

Gupta, N. D. and Mazurov, V. D. (1999).
On groups with small orders of elements.
Bull. Austral. Math. Soc., 60:197–205.

$\mu(G) = \{5, 6\}$

[Mazurov and Mamontov, 2009]

Let $\mu(G) = \{5, 6\}$. Then G is a soluble locally finite group and one of the following conditions holds:

- G is an extension of elementary Abelian 5-group by C_6;
- G is an extension of a 3-group of nilpotency class 3 by D_{10};
- G is an extension of the direct product of a 3-group of nilpotency class 2 and an elementary Abelian 2-group by a group of order 5.

Mazurov, V. D. and Mamontov, A. S. (2009).
On periodic groups with small orders of elements.
Sib. Math. J., 50(2):316–321.

$\mu(G) = \{3, 4, 5\}$

[Mazurov, 2000]

If $\mu(G) = \{3, 4, 5\}$, then G is locally finite and either $G \simeq A_6$, or $G = VC$, where V is a non-trivial elementary Abelian normal 2-subgroup, and $C \simeq A_5$.

Mazurov, V. D. (2000).
Groups of exponent 60 with prescribed orders of elements.
Algebra and Logic, 39(3):189–198.

$\mu(G) = \{3, 8\}$

[Mazurov, 2010]

Let G be a group with $\mu(G) = \{3, 8\}$. Then G is lf and one of the fc holds:

- $G = VQ$ where V is a non-trivial normal eA 3-subgroup and Q is a 2-subgroup acting freely on V and isomorphic to either C_8 or Q_{16};
- $G = T\langle a\rangle$ where T is a normal 2-subgroup which is nilpotent of class 2 and $|a| = 3$;
- $G = TS$ where T is a normal 2-subgroup which is nilpotent of class 2 and exponent 4 and $S \simeq S_3$.

In particular, G is soluble of length at most 4.

Mazurov, V. D. (2010).
Groups of exponent 24.
Algebra and Logic, 49(6):515–525.

[Lytkina et al., 2012]

Let G be a group of period 12, such that the order of the product of every two involutions is distinct from 6, and let H be the subgroup generated by all involutions of G. Then G is lf and one of the fc holds.

- G is a 3-group.
- H is an extension of a 3-group by a group of order 2, and G is a split extension of a 3-group by some non-trivial subgroup of Q_8 or $SL_2(3)$.
- H is isomorphic to a split extension of elementary Abelian 2-group V by a non-abelian group of order 6, acting faithfully on V, and G/H is a 3-group.
- $G/O_2(G)$, where $O_2(G)$ is the largest normal 2-subgroup of G, is an extension of a 3-group by an elementary Abelian 2-group, and $H \leq O_2(G)$.

Lytkina, D. V., Mazurov, V. D., and Mamontov, A. S. (2012).
On local finiteness of some groups of period 12.
Siberian mathematical journal, 53(6):1105–1109.

Groups of period 12

[Mamontov and Mazurov, 2013]

Let G be a group of period 12, such that the order of the product of every two involutions is distinct from 4. Then G is locally finite.

Mamontov, A. S. and Mazurov, V. D. (2013).
Involutions in groups of exponent 12.
Algebra and Logic, 52(1):67–71.

Groups of period 12

[Mamontov, 2013]

Groups of period 12 without elements of order 12 are locally finite.

Mamontov, A. S. (2013).
Groups of exponent 12 without elements of order 12.
Siberian Mathematical Journal, 54(1):114–118.

$\mu(G) = \{3, 4, 7\}$

[Lytkina and Kuznetsov, 2007]

If $\mu(G) = \{3, 4, 7\}$, then $G \simeq L_2(7)$.

Lytkina, D. V. and Kuznetsov, A. A. (2007).
Recognizability by spectrum of the group $l_2(7)$ in the class of all groups.
Sib. Electronic Math. Reports, 4:300–303.