Four classes of verbal subgroups

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So a verbal subgroup V is VN-verbal iff it contains $\gamma_c(\hat{F}^e)$.

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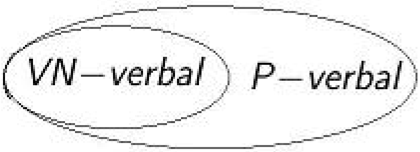

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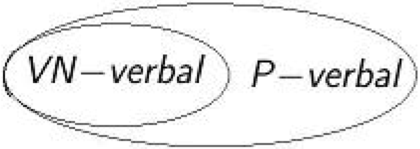

- The inclusion is proper: there are infinite Burnside groups and examples by A. Yu. Ol'shanskii and A. Storozhev.

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$$a\left(u(x,y), v(x,y)\right) \equiv b\left(u(x,y), v(x,y)\right),$$

because $modV_2$ it has is $a(u,u) \equiv b(u,u)$ and hence $u^k \equiv u^k$. □

R-verbal subgroups

R-property of V says that $(F/V)'$ is finitely generated (for $n < \infty$)

Criterion for R-verbal subgroups

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V is R-verbal iff $\exists m \in N$, such that F/V satisfies a law

$$[x, {}_m y] \equiv u(x,y), \qquad u(x,y) \in \langle x, [x,y], [x, {}_2 y], \ldots [x, {}_{m-1} y] \rangle.$$

Proof is long.

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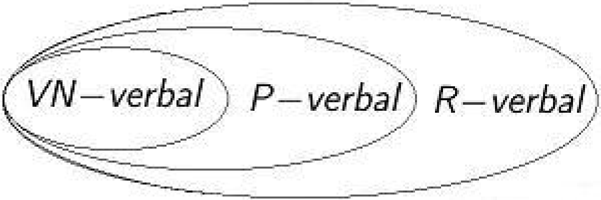

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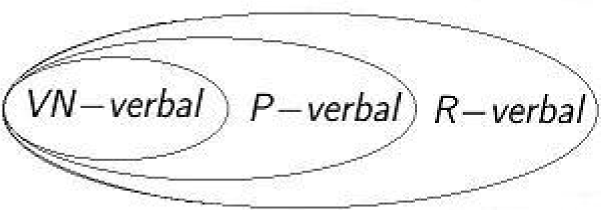

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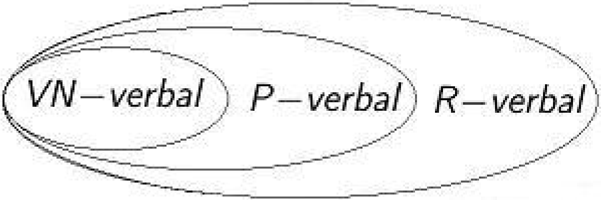

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Will the n-Engel laws prove that the inclusion is proper?

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V is called M-verbal because a law which is not satisfied in any of $\mathfrak{A}_p\mathfrak{A}$ is called *Milnor identity* (F.Point 1996).

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Hence F/VF'' satisfies a positive law, which implies (*). □

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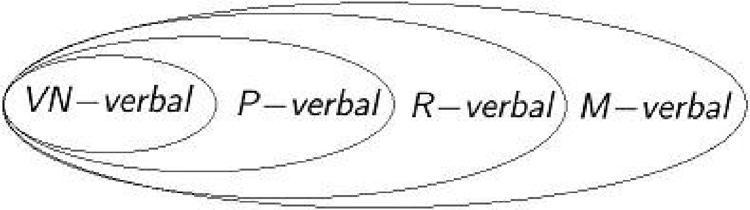

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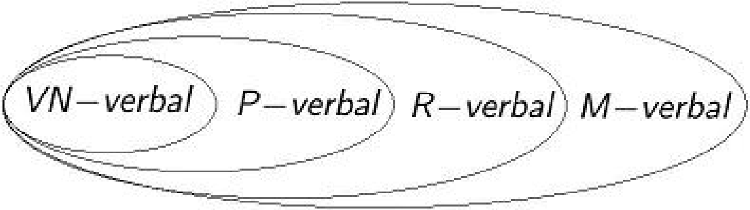

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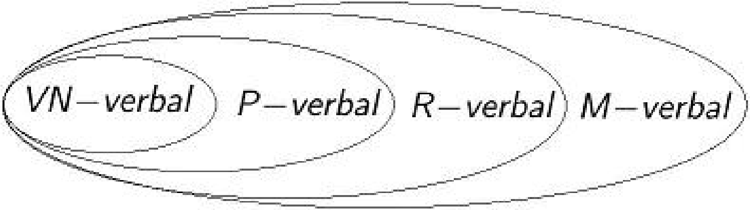

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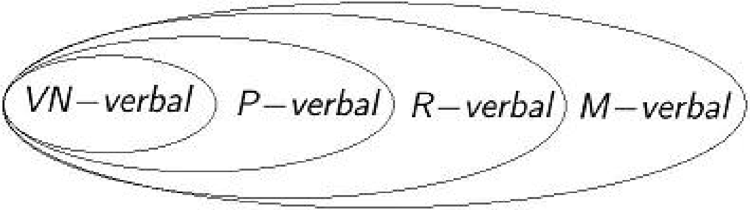

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